Supplementary Material of "Tensor Wheel Decomposition: Theory and Application to Tensor Completion"

Zhong-Cheng Wu, Liang-Jian Deng, *Senior Member, IEEE*, Ting-Zhu Huang, *Member, IEEE*, Hong-Xia Dou, Gemine Vivone, *Senior Member, IEEE*, Yu Liu, *Senior Member, IEEE*

Abstract

The supplementary material is structured as follows. Appendix A recapitulates the definitions of the generalized tensor k-unfolding and the generalized tensor k-contraction and further presents the detailed proofs for the algebraic properties of the generalized tensor k-contraction. Appendix B supplements the additional analysis of the tensor-form TW decomposition and gives the proofs of Theorems 1-4 in the main text. Appendix C explores a corollary of the core-connected invariance theorem and completes the theoretical proof of Proposition 1 in the main text. Appendix D introduces the main preliminaries regarding the truncated SVD and establishes the approximation error bound of the TW-SVD algorithm. Appendix E formats the proof of the local convergence theorem and provides the proofs of Lemmas 1-4 in the main text. Finally, Appendix F verifies the numerical convergence of the proposed PAM-based Algorithm 3 in the main text.

CONTENTS

This work was supported in part by the Fundamental and Interdisciplinary Disciplines Breakthrough Plan of the Ministry of Education of China under Grant JYB2025XDXM109, in part by the National Natural Science Foundation of China under Grant 62576132 and Grant U23A20294, and in part by the Project of the Department of Science and Technology of Sichuan Province under Grant 2025YFNH0001. *(Corresponding authors: Liang-Jian Deng; Yu Liu.)*

Z.-C. Wu and Y. Liu are with the School of Instrument Science and Opto-electronics Engineering, Hefei University of Technology, Hefei 230009, China (e-mail: zhongcheng.wu@hfut.edu.cn; yuliu@hfut.edu.cn).

L.-J. Deng and T.-Z. Huang are with the School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China (e-mail: liangjian.deng@uestc.edu.cn; tingzhuhuang@126.com).

H.-X. Dou is with the School of Science, Xihua University, Chengdu 610039, China (e-mail: hongxia.dou@mail.xhu.edu.cn).

G. Vivone is with the Institute of Methodologies for Environmental Analysis (CNR-IMAA), Tito Scalo 85050, Italy (e-mail: gemine.vivone@imaa.cnr.it).

Appendix A
Algebraic Properties of Generalized Tensor k-Contraction

The generalized k-contraction between two proper tensors performs the multi-linear product, which essentially generalizes the traditional matrix product. Therefore, the algebraic properties of the generalized tensor k-contraction are studied by analogy with those of matrix product. To facilitate the presentation, Definitions 2-3 of the main text are first replicated in Appendix A-A.

A. Formulations of Definitions 2-3 in the main text

Definition A.1 (Generalized Tensor k-Unfolding). *Let $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ be a reordering of vector $(1, 2, \cdots, N)$, then for an Nth-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, the generalized k-unfolding $(0 \le k \le N,\ k \in \mathbb{Z})$ of $\mathcal{X}$ is defined as a matrix $\mathbf{X}_{[\boldsymbol{n};k]} \in \mathbb{R}^{\prod_{i=1}^{k} I_{n_i} \times \prod_{j=k+1}^{N} I_{n_j}}$, which requires*

$$\mathbf{X}_{[\boldsymbol{n};k]}(\overline{i_{n_1} i_{n_2} \cdots i_{n_k}}, \overline{i_{n_{k+1}} \cdots i_{n_N}}) = \vec{\mathcal{X}}^{\boldsymbol{n}}\left(i_{n_1}, i_{n_2}, \cdots, i_{n_N}\right), \tag{1}$$

where $\vec{\mathcal{X}}^{\boldsymbol{n}}$ is the $\boldsymbol{n}$-based tensor permutation of $\mathcal{X}$, and the multi-indices $\overline{i_{n_1} i_{n_2} \cdots i_{n_k}}$ and $\overline{i_{n_{k+1}} i_{n_{k+2}} \cdots i_{n_N}}$ are defined by $1 + \sum_{i=1}^{k}(i_{n_i} - 1) \prod_{j=1}^{i-1} I_{n_j}$ and $1 + \sum_{i=k+1}^{N}(i_{n_i} - 1) \prod_{j=k+1}^{i-1} I_{n_j}$, respectively. When k is 0 and N, $\mathbf{x}_{[\boldsymbol{n};0]} \in \mathbb{R}^{1 \times \prod_{j=1}^{N} I_{n_j}}$ and $\mathbf{x}_{[\boldsymbol{n};N]} \in \mathbb{R}^{\prod_{i=1}^{N} I_{n_i} \times 1}$ imply two generalized vectorizations. Conversely, the inverse operator of the k-unfolding yields $\mathcal{X} = \mathtt{Fold}_{[\boldsymbol{n};k]}(\mathbf{X}_{[\boldsymbol{n};k]})$ or $\mathcal{X} = \mathtt{Fold}_{[\boldsymbol{n};k]}(\mathbf{x}_{[\boldsymbol{n};k]})$ for $k = 0, N$.

Definition A.2 (Generalized Tensor k-Contraction). *Given an Mth-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_M}$ and an Nth-order tensor $\mathcal{Y} \in \mathbb{R}^{J_1 \times J_2 \times \cdots \times J_N}$ with k common modes $(1 \le k \le M \wedge N,\ k \in \mathbb{Z})$. Assume that two vectors $\boldsymbol{m} = (m_1, m_2, \cdots, m_M)$ and $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ respectively indicate the reordering of vectors $(1, 2, \cdots, M)$ and $(1, 2, \cdots, N)$, satisfying $I_{m_i} = J_{n_i}$ for $i = 1, 2, \cdots, k$, $m_{k+1} < m_{k+2} < \cdots < m_M$ and $n_{k+1} < n_{k+2} < \cdots < n_N$. Then the generalized k-contraction between $\mathcal{X}$ and $\mathcal{Y}$ along the k modes specifies an $(M + N - 2k)$th-order tensor $\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y} \in \mathbb{R}^{I_{m_{k+1}} \times \cdots \times I_{m_M} \times J_{n_{k+1}} \times \cdots \times J_{n_N}}$, which is given as follows,*

$$\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y} = \mathtt{Fold}_{[(1:M+N-2k);M-k]}\big(\mathbf{X}_{[\boldsymbol{m};k]}^{\top} \mathbf{Y}_{[\boldsymbol{n};k]}\big), \tag{2}$$

where $\mathbf{X}_{[\boldsymbol{m};k]}$ and $\mathbf{Y}_{[\boldsymbol{n};k]}$ are the $\boldsymbol{m}$-based and $\boldsymbol{n}$-based k-unfoldings of tensors $\mathcal{X}$ and $\mathcal{Y}$, respectively. When $k = M \wedge N$, the lowercase symbols, i.e., $\mathbf{x}_{[\boldsymbol{m};k]}$ and $\mathbf{y}_{[\boldsymbol{n};k]}$, can be aptly adopted.

B. Proof of Property 1

Property A.1. *Let $\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y}$ be the k-contracted tensor in Definition A.2, then the $(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N})$-th element of $\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y}$ can be computed by*

$$\begin{aligned}(\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y})(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N}) = &\sum_{i_{m_1}=j_{n_1}=1}^{I_{m_1}} \sum_{i_{m_2}=j_{n_2}=1}^{I_{m_2}} \cdots \sum_{i_{m_k}=j_{n_k}=1}^{I_{m_k}} \\ &\big(\mathcal{X}(i_1, i_2, \cdots, i_M)\mathcal{Y}(j_1, j_2, \cdots, j_N)\big).\end{aligned} \tag{3}$$

Proof. According to the above Definition A.2, we firstly have

$$\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y} = \mathtt{Fold}_{[(1:M+N-2k);M-k]}\big(\mathbf{X}_{[\boldsymbol{m};k]}^{\top} \mathbf{Y}_{[\boldsymbol{n};k]}\big), \tag{4}$$

which is equivalent to

$$(\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y})_{[(1:M+N-2k);M-k]} = \mathbf{X}_{[\boldsymbol{m};k]}^{\top} \mathbf{Y}_{[\boldsymbol{n};k]}. \tag{5}$$

Following the above Definition A.1, the $(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N})$-th element of the tensor $\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y}$, which is of size $I_{m_{k+1}} \times \cdots \times I_{m_M} \times J_{n_{k+1}} \times \cdots \times J_{n_N}$, satisfies

$$\begin{aligned}(\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y})(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N}) &= (\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y})_{[(1:M+N-2k);M-k]}(\overline{i_{m_{k+1}} \cdots i_{m_M}}, \overline{j_{n_{k+1}} \cdots j_{n_N}}) \\ &= \sum_{r=1}^{\prod_{i=1}^{k} I_{m_i}} \mathbf{X}_{[\boldsymbol{m};k]}^{\top}(\overline{i_{m_{k+1}} \cdots i_{m_M}}, r)\mathbf{Y}_{[\boldsymbol{n};k]}(r, \overline{j_{n_{k+1}} \cdots j_{n_N}}) \\ &= \sum_{r=1}^{\prod_{i=1}^{k} I_{m_i}} \mathbf{X}_{[\boldsymbol{m};k]}(r, \overline{i_{m_{k+1}} \cdots i_{m_M}})\mathbf{Y}_{[\boldsymbol{n};k]}(r, \overline{j_{n_{k+1}} \cdots j_{n_N}}),\end{aligned} \tag{6}$$

where $\overline{i_{m_{k+1}} \cdots i_{m_M}}$ and $\overline{j_{n_{k+1}} \cdots j_{n_N}}$ are defined by $1 + \sum_{i=k+1}^{M}(i_{m_i} - 1) \prod_{j=k+1}^{i-1} I_{m_j}$ and $1 + \sum_{i=k+1}^{N}(j_{n_i} - 1) \prod_{j=k+1}^{i-1} J_{n_j}$, respectively. From Definition A.1 again, we can get

$$\mathbf{X}_{[\boldsymbol{m};k]}(\overline{i_{m_1} \cdots i_{m_k}}, \overline{i_{m_{k+1}} \cdots i_{m_M}}) = \vec{\mathcal{X}}^{\boldsymbol{m}}(i_{m_1}, i_{m_2}, \cdots, i_{m_k}, i_{m_{k+1}}, \cdots, i_{m_M}), \tag{7}$$

and

$$\mathbf{Y}_{[\boldsymbol{n};k]}(\overline{j_{n_1}\cdots j_{n_k}}, \overline{j_{n_{k+1}}\cdots j_{n_N}}) = \vec{\mathcal{Y}}^{\boldsymbol{n}}(j_{n_1}, j_{n_2}, \cdots, j_{n_k}, j_{n_{k+1}}, \cdots, j_{n_N}), \tag{8}$$

where $\overline{i_{m_1}\cdots i_{m_k}}$ and $\overline{j_{n_1}\cdots j_{n_k}}$ are defined by $1+\sum_{i=1}^{k}(i_{m_i}-1)\prod_{j=1}^{i-1} I_{m_j}$ and $1+\sum_{i=1}^{k}(j_{n_i}-1)\prod_{j=1}^{i-1} J_{n_j}$, respectively. Since $I_{m_i} = J_{n_i},\ i = 1, 2, \cdots, k$, the equality $\overline{i_{m_1}\cdots i_{m_k}} = \overline{j_{n_1}\cdots j_{n_k}}$ generates

$$i_{m_i} = j_{n_i},\ i = 1, 2, \cdots, k. \tag{9}$$

Let $r_i = i_{m_i} = j_{n_i},\ i = 1, 2, \cdots, k$, we accordingly deduce

$$\begin{aligned}(\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y})(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N}) &= \sum_{r=1}^{\prod_{i=1}^{k} I_{m_i}} \mathbf{X}_{[\boldsymbol{m};k]}(r, \overline{i_{m_{k+1}}\cdots i_{m_M}})\mathbf{Y}_{[\boldsymbol{n};k]}(r, \overline{j_{n_{k+1}}\cdots j_{n_N}}) \\ &= \sum_{r_1}^{I_{m_1}}\sum_{r_2}^{I_{m_2}}\cdots\sum_{r_k}^{I_{m_k}} \Big(\vec{\mathcal{X}}^{\boldsymbol{m}}(r_1, r_2, \cdots, r_k, i_{m_{k+1}}, \cdots, i_{m_M}) \\ &\qquad \vec{\mathcal{Y}}^{\boldsymbol{n}}(r_1, r_2, \cdots, r_k, j_{n_{k+1}}, \cdots, j_{n_N})\Big) \\ &= \sum_{i_{m_1}=j_{n_1}=1}^{I_{m_1}}\ \sum_{i_{m_2}=j_{n_2}=1}^{I_{m_2}} \cdots \sum_{i_{m_k}=j_{n_k}=1}^{I_{m_k}} \\ &\qquad \big(\mathcal{X}(i_1, i_2, \cdots, i_M)\mathcal{Y}(j_1, j_2, \cdots, j_N)\big),\end{aligned} \tag{10}$$

where vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ are pre-defined before contraction. The proof is completed. □

C. Proof of Property 2

Property A.2. *Given an Mth-order tensor $\mathcal{X} \in \mathbb{R}^{I_1\times I_2\times\cdots\times I_M}$ and an Nth-order tensor $\mathcal{Y} \in \mathbb{R}^{J_1\times J_2\times\cdots\times J_N}$ with k common modes ($1 \le k \le M \wedge N,\ k \in \mathbb{Z}$). Assume that two vectors $\boldsymbol{m} = (m_1, m_2, \cdots, m_M)$ and $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ respectively indicate the reordering of vectors $(1, 2, \cdots, M)$ and $(1, 2, \cdots, N)$, satisfying $I_{m_i} = J_{n_i}$ for $i = 1, 2, \cdots, k$, $m_{k+1} < m_{k+2} < \cdots < m_M$ and $n_{k+1} < n_{k+2} < \cdots < n_N$. Then the computation of the generalized k-contraction imposes*

$$\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y} = \overrightarrow{(\mathcal{Y} \times_{m_1,m_2,\cdots,m_k}^{n_1,n_2,\cdots,n_k} \mathcal{X})}^{\boldsymbol{v}}, \tag{11}$$

where $\boldsymbol{v} = (N-k+1 : M+N-2k, 1 : N-k)$.

Proof. Following the proof of Property A.1, the $(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N})$-th element of the tensor $\mathcal{X} \times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k} \mathcal{Y} \in \mathbb{R}^{I_{m_{k+1}}\times\cdots\times I_{m_M}\times J_{n_{k+1}}\times\cdots\times J_{n_N}}$ is computed by

$$\begin{aligned}(\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y})(i_{m_{k+1}}, \cdots, i_{m_M}, j_{n_{k+1}}, \cdots, j_{n_N}) &= (\mathcal{X} \times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k} \mathcal{Y})_{[(1:M+N-2k);M-k]}(\overline{i_{m_{k+1}}\cdots i_{m_M}}, \overline{j_{n_{k+1}}\cdots j_{n_N}}) \\ &= \sum_{r=1}^{\prod_{i=1}^{k} I_{m_i}} \mathbf{X}_{[\boldsymbol{m};k]}^{\top}(\overline{i_{m_{k+1}}\cdots i_{m_M}}, r)\mathbf{Y}_{[\boldsymbol{n};k]}(r, \overline{j_{n_{k+1}}\cdots j_{n_N}}) \\ &= \sum_{r=1}^{\prod_{i=1}^{k} I_{m_i}} \mathbf{X}_{[\boldsymbol{m};k]}(r, \overline{i_{m_{k+1}}\cdots i_{m_M}})\mathbf{Y}_{[\boldsymbol{n};k]}(r, \overline{j_{n_{k+1}}\cdots j_{n_N}}),\end{aligned} \tag{12}$$

where $\overline{i_{m_{k+1}}\cdots i_{m_M}}$ and $\overline{j_{n_{k+1}}\cdots j_{n_N}}$ are defined by $1+\sum_{i=k+1}^{M}(i_{m_i}-1)\prod_{j=k+1}^{i-1} I_{m_j}$ and $1+\sum_{i=k+1}^{N}(j_{n_i}-1)\prod_{j=k+1}^{i-1} J_{n_j}$, respectively. Similarly, the generalized tensor k-contraction also provides

$$\mathcal{Y} \times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k} \mathcal{X} = \mathtt{Fold}_{[(1:M+N-2k);N-k]}\big(\mathbf{Y}_{[\boldsymbol{n};k]}^{\top}\mathbf{X}_{[\boldsymbol{m};k]}\big) \Longleftrightarrow (\mathcal{Y} \times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k} \mathcal{X})_{[(1:M+N-2k);N-k]} = \mathbf{Y}_{[\boldsymbol{n};k]}^{\top}\mathbf{X}_{[\boldsymbol{m};k]}. \tag{13}$$

Therefore, the $(j_{n_{k+1}},\cdots,j_{n_N},i_{m_{k+1}},\cdots,i_{m_M})$-th element of the tensor $\mathcal{Y}\times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k}\mathcal{X}\in\mathbb{R}^{J_{n_{k+1}}\times\cdots\times J_{n_N}\times I_{m_{k+1}}\times\cdots\times I_{m_M}}$ satisfies

$$\begin{aligned}(\mathcal{Y}\times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k}\mathcal{X})(j_{n_{k+1}},\cdots,j_{n_N},i_{m_{k+1}},\cdots,i_{m_M}) &= (\mathcal{Y}\times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k}\mathcal{X})_{[(1:M+N-2k);N-k]}(\overline{j_{n_{k+1}}\cdots j_{n_N}},\overline{i_{m_{k+1}}\cdots i_{m_M}})\\ &= \sum_{r=1}^{\prod_{i=1}^{k}I_{m_i}}\mathbf{Y}_{[\boldsymbol{n};k]}^{\top}(\overline{j_{n_{k+1}}\cdots j_{n_N}},r)\mathbf{X}_{[\boldsymbol{m};k]}(r,\overline{i_{m_{k+1}}\cdots i_{m_M}})\\ &= \sum_{r=1}^{\prod_{i=1}^{k}I_{m_i}}\mathbf{Y}_{[\boldsymbol{n};k]}(r,\overline{j_{n_{k+1}}\cdots j_{n_N}})\mathbf{X}_{[\boldsymbol{m};k]}(r,\overline{i_{m_{k+1}}\cdots i_{m_M}})\\ &= \sum_{r=1}^{\prod_{i=1}^{k}I_{m_i}}\mathbf{X}_{[\boldsymbol{m};k]}(r,\overline{i_{m_{k+1}}\cdots i_{m_M}})\mathbf{Y}_{[\boldsymbol{n};k]}(r,\overline{j_{n_{k+1}}\cdots j_{n_N}}).\end{aligned} \tag{14}$$

From formulas (12) and (14), we numerically have

$$(\mathcal{X}\times_{n_1,\cdots,n_k}^{m_1,\cdots,m_k}\mathcal{Y})(i_{m_{k+1}},\cdots,i_{m_M},j_{n_{k+1}},\cdots,j_{n_N})=(\mathcal{Y}\times_{m_1,\cdots,m_k}^{n_1,\cdots,n_k}\mathcal{X})(j_{n_{k+1}},\cdots,j_{n_N},i_{m_{k+1}},\cdots,i_{m_M}). \tag{15}$$

Based on Definition 1 (i.e., tensor permutation) of the main text, when vector $\boldsymbol{v}=(N-k+1:M+N-2k,1:N-k)$, the above formula (15) directly indicates

$$\mathcal{X}\times_{n_1,n_2,\cdots,n_k}^{m_1,m_2,\cdots,m_k}\mathcal{Y}=\overrightarrow{(\mathcal{Y}\times_{m_1,m_2,\cdots,m_k}^{n_1,n_2,\cdots,n_k}\mathcal{X})}^{\boldsymbol{v}}. \tag{16}$$

The proof is completed. □

D. Proof of Property 3

Property A.3. *Given an Mth-order tensor $\mathcal{X}\in\mathbb{R}^{I_1\times I_2\times\cdots\times I_M}$, an Nth-order tensor $\mathcal{Y}\in\mathbb{R}^{J_1\times J_2\times\cdots\times J_N}$, and an Hth-order tensor $\mathcal{Z}\in\mathbb{R}^{L_1\times L_2\times\cdots\times L_H}$, where $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{Y}$ and $\mathcal{Z}$, and $\mathcal{X}$ and $\mathcal{Z}$ have k_1, k_2, and k_3 ($2\le k_1+k_3\le M$, $2\le k_1+k_2\le N$, $2\le k_2+k_3\le H$, $k_1,k_2,k_3\in\mathbb{N}^+$) common modes, respectively. Assume that three vectors $\boldsymbol{m}=(m_1,m_2,\cdots,m_M)$, $\boldsymbol{n}=(n_1,n_2,\cdots,n_N)$, and $\boldsymbol{h}=(h_1,h_2,\cdots,h_H)$ respectively indicate the reordering of vectors $(1,2,\cdots,M)$, $(1,2,\cdots,N)$, and $(1,2,\cdots,H)$, such that $I_{m_i}=J_{n_i}$ for $i=1,2,\cdots,k_1$, $J_{n_{k_1+i}}=L_{h_{k_3+i}}$ for $i=1,2,\cdots,k_2$, $I_{m_{k_1+i}}=L_{h_i}$ for $i=1,2,\cdots,k_3$, $m_{k_1+k_3+1}<m_{k_1+k_3+2}<\cdots<m_M$, $n_{k_1+k_2+1}<n_{k_1+k_2+2}<\cdots<n_N$, and $h_{k_2+k_3+1}<h_{k_2+k_3+2}<\cdots<h_H$. When the contracted modes are fully and properly specified, the successively contracted tensors with different precedences are equivalent as follows,*

$$\mathcal{X}\times_{1,2,\cdots,k_1,N-k_2+1,\cdots,N-k_2+k_3}^{m_1,m_2,\cdots,m_{k_1},m_{k_1+1},\cdots,m_{k_1+k_3}}(\mathcal{Y}\times_{h_{k_3+1},\cdots,h_{k_3+k_2}}^{n_{k_1+1},\cdots,n_{k_1+k_2}}\mathcal{Z})=(\mathcal{X}\times_{n_1,n_2,\cdots,n_{k_1}}^{m_1,m_2,\cdots,m_{k_1}}\mathcal{Y})\times_{h_1,h_2,\cdots,h_{k_3},h_{k_3+1},\cdots,h_{k_3+k_2}}^{1,2,\cdots,k_3,M-k_1+1,\cdots,M-k_1+k_2}\mathcal{Z}. \tag{17}$$

Proof. Relying upon Definition A.2, we can get that both the tensor

$$(\mathcal{X}\times_{n_1,\cdots,n_{k_1}}^{m_1,\cdots,m_{k_1}}\mathcal{Y})\times_{h_1,h_2,\cdots,h_{k_3},h_{k_3+1},\cdots,h_{k_3+k_2}}^{1,2,\cdots,k_3,M-k_1+1,\cdots,M-k_1+k_2}\mathcal{Z} \tag{18}$$

and the tensor

$$\mathcal{X}\times_{1,2\cdots,k_1,N-k_2+1,\cdots,N-k_2+k_3}^{m_1,m_2\cdots,m_{k_1},m_{k_1+1},\cdots,m_{k_1+k_3}}(\mathcal{Y}\times_{h_{k_3+1},\cdots,h_{k_3+k_2}}^{n_{k_1+1},\cdots,n_{k_1+k_2}}\mathcal{Z}) \tag{19}$$

are of size $I_{m_{k_1+k_3+1}}\times\cdots\times I_{m_M}\times J_{n_{k_1+k_2+1}}\times\cdots\times J_{n_N}\times L_{h_{k_2+k_3+1}}\times\cdots\times L_{h_H}$. From Property A.1, we further have

$$\begin{aligned}&\Big((\mathcal{X}\times_{n_1,\cdots,n_{k_1}}^{m_1,\cdots,m_{k_1}}\mathcal{Y})\times_{h_1,h_2,\cdots,h_{k_3},h_{k_3+1},\cdots,h_{k_3+k_2}}^{1,2,\cdots,k_3,M-k_1+1,\cdots,M-k_1+k_2}\mathcal{Z}\Big)(i_{m_{k_1+k_3+1}},\cdots,i_{m_M},j_{n_{k_1+k_2+1}},\cdots,j_{n_N},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\\ &=\sum_{r_1=1}^{L_{h_1}}\sum_{r_2=1}^{L_{h_2}}\cdots\sum_{r_{k_2+k_3}=1}^{L_{h_{k_2+k_3}}}\Big((\mathcal{X}\times_{n_1,\cdots,n_{k_1}}^{m_1,\cdots,m_{k_1}}\mathcal{Y})(r_1,\cdots,i_{m_M},r_{k_3+1},\cdots,j_{n_N})\vec{\mathcal{Z}}^{\boldsymbol{h}}(r_1,r_2,\cdots,r_{k_2+k_3},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\Big)\\ &=\sum_{r_1=1}^{L_{h_1}}\sum_{r_2=1}^{L_{h_2}}\cdots\sum_{r_{k_2+k_3}=1}^{L_{h_{k_2+k_3}}}\sum_{r_{k_2+k_3+1}=1}^{I_{m_1}}\cdots\sum_{r_{k_1+k_2+k_3}=1}^{I_{m_{k_1}}}\Big(\vec{\mathcal{X}}^{\boldsymbol{m}}(r_{k_2+k_3+1},\cdots,r_{k_1+k_2+k_3},r_1,\cdots,r_{k_3},i_{m_{k_1+k_3+1}},\cdots,i_{m_M})\\ &\qquad\vec{\mathcal{Y}}^{\boldsymbol{n}}(r_{k_2+k_3+1},\cdots,r_{k_1+k_2+k_3},r_{k_3+1},\cdots,r_{k_2+k_3},j_{n_{k_1+k_2+1}},\cdots,j_{n_N})\\ &\qquad\vec{\mathcal{Z}}^{\boldsymbol{h}}(r_1,r_2,\cdots,r_{k_2+k_3},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\Big).\end{aligned} \tag{20}$$

Similarly, we also have

$$\begin{aligned}
&\Big(\mathcal{X}\times^{m_1,m_2\cdots,m_{k_1},m_{k_1+1},\cdots,m_{k_1+k_3}}_{1,2\cdots,k_1,N-k_2+1,\cdots,N-k_2+k_3}(\mathcal{Y}\times^{n_{k_1+1},\cdots,n_{k_1+k_2}}_{h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z})\Big)(i_{m_{k_1+k_3+1}},\cdots,i_{m_M},j_{n_{k_1+k_2+1}},\cdots,j_{n_N},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\\
&=\sum_{r_1=1}^{I_{m_1}}\sum_{r_2=1}^{I_{m_2}}\cdots\sum_{r_{k_1+k_3}=1}^{I_{m_{k_1+k_3}}}\Big(\vec{\mathcal{X}}^{\boldsymbol{m}}(r_1,r_2,\cdots,r_{k_1+k_3},i_{m_{k_1+k_3+1}},\cdots,i_{m_M})(\mathcal{Y}\times^{n_{k_1+1},\cdots,n_{k_1+k_2}}_{h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z})(r_1,\cdots,j_{n_N},r_{k_1+1},\cdots,l_{h_H})\Big)\\
&=\sum_{r_1=1}^{I_{m_1}}\sum_{r_2=1}^{I_{m_2}}\cdots\sum_{r_{k_1+k_3}=1}^{I_{m_{k_1+k_3}}}\sum_{r_{k_1+k_3+1}=1}^{L_{h_{k_3+1}}}\cdots\sum_{r_{k_1+k_2+k_3}=1}^{L_{h_{k_2+k_3}}}\Big(\vec{\mathcal{X}}^{\boldsymbol{m}}(r_1,r_2,\cdots,r_{k_1+k_3},i_{m_{k_1+k_3+1}},\cdots,i_{m_M})\\
&\qquad\vec{\mathcal{Y}}^{\boldsymbol{n}}(r_1,\cdots,r_{k_1},r_{k_1+k_3+1},\cdots,r_{k_1+k_2+k_3},j_{n_{k_1+k_2+1}},\cdots,j_{n_N})\\
&\qquad\vec{\mathcal{Z}}^{\boldsymbol{h}}(r_{k_1+1},\cdots,r_{k_1+k_3},r_{k_1+k_3+1},\cdots,r_{k_1+k_2+k_3},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\Big).
\end{aligned}\tag{21}$$

By comparing formulas (20) and (21), we essentially have

$$\begin{aligned}
&\Big(\mathcal{X}\times^{m_1,m_2\cdots,m_{k_1},m_{k_1+1},\cdots,m_{k_1+k_3}}_{1,2\cdots,k_1,N-k_2+1,\cdots,N-k_2+k_3}(\mathcal{Y}\times^{n_{k_1+1},\cdots,n_{k_1+k_2}}_{h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z})\Big)(i_{m_{k_1+k_3+1}},\cdots,i_{m_M},j_{n_{k_1+k_2+1}},\cdots,j_{n_N},l_{h_{k_2+k_3+1}},\cdots,l_{h_H})\\
&\quad=\Big((\mathcal{X}\times^{m_1,\cdots,m_{k_1}}_{n_1,\cdots,n_{k_1}}\mathcal{Y})\times^{1,2,\cdots,k_3,M-k_1+1,\cdots,M-k_1+k_2}_{h_1,h_2,\cdots,h_{k_3},h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z}\Big)(i_{m_{k_1+k_3+1}},\cdots,i_{m_M},j_{n_{k_1+k_2+1}},\cdots,j_{n_N},l_{h_{k_2+k_3+1}},\cdots,l_{h_H}),
\end{aligned}\tag{22}$$

which is equivalent to

$$(\mathcal{X}\times^{m_1,\cdots,m_{k_1}}_{n_1,\cdots,n_{k_1}}\mathcal{Y})\times^{1,2,\cdots,k_3,M-k_1+1,\cdots,M-k_1+k_2}_{h_1,h_2,\cdots,h_{k_3},h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z}=\mathcal{X}\times^{m_1,m_2\cdots,m_{k_1},m_{k_1+1},\cdots,m_{k_1+k_3}}_{1,2\cdots,k_1,N-k_2+1,\cdots,N-k_2+k_3}(\mathcal{Y}\times^{n_{k_1+1},\cdots,n_{k_1+k_2}}_{h_{k_3+1},\cdots,h_{k_3+k_2}}\mathcal{Z}).\tag{23}$$

This completes the proof. □

Properties A.1-A.3 are formally easily comprehensible and provide great help for subsequent proofs, see Appendix B.

Appendix B
Analysis of TW Model and Proofs of Theorems 1-4

A. Analysis for TW Model

Let $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ be an Nth-order tensor, then the proposed TW decomposition numerically establishes the element-wise relationship as follows,

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big), \tag{24}$$

where $\mathcal{G}_k \in \mathbb{R}^{R_k \times I_k \times L_k \times R_{k+1}}$, $k = 1, 2, \cdots, N$, and $\mathcal{C} \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_k \times \cdots \times L_N}$ are the corresponding $N+1$ TW factors. Although model (24) is mathematically more intuitive, it is not suitable for practical computation and theoretical analysis. Accordingly, we further provide the following tensor form of TW decomposition in the main text as

$$\mathcal{X} = \mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C}, \tag{25}$$

which is given by only relying upon the wheel topology with tensor contractions. For more rigor, we supplement the proof.

Proof. According to formula (24), the proposed TW decomposition yields

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big). \tag{26}$$

From Property A.1, we thus induce

$$\begin{aligned}
\mathcal{X}(i_1, i_2, \cdots, i_N) &= \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \\
&\qquad \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big) \\
&= \sum_{r_1=1}^{R_1} \sum_{r_3=1}^{R_3} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \left(\sum_{r_2=1}^{R_2} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \Big) \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \right. \\
&\qquad \left. \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \right) \\
&= \sum_{r_1=1}^{R_1} \sum_{r_3=1}^{R_3} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big((\mathcal{G}_1 \times_1^4 \mathcal{G}_2)(r_1, i_1, l_1, i_2, l_2, r_3) \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \\
&\qquad \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big) \\
&\;\;\vdots \\
&= \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \left(\sum_{r_1=1}^{R_1} \sum_{r_N=1}^{R_N} \Big((\mathcal{G}_1 \times_1^4 \cdots \times_1^{2N-2} \mathcal{G}_{N-1})(r_1, i_1, l_1, \cdots, l_{N-1}, r_N) \mathcal{G}_N(r_N, i_N, l_N, r_1) \Big) \right. \\
&\qquad \left. \mathcal{C}(l_1, l_2, \cdots, l_N) \right) \\
&= \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big((\mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N)(i_1, l_1, \cdots, i_N, l_N) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big) \\
&= (\mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C})(i_1, i_2, \cdots, i_N),
\end{aligned} \tag{27}$$

which naturally confirms $\mathcal{X} = \mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C}$. The proof is completed. □

B. Proof of Theorem 1

Theorem B.1 (Core-Centered Circular Invariance). *Given an Nth-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ and its TW decomposition TW$[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!]$. Assume that $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is the circular reordering of vector $(1, 2, \cdots, N)$, then the core-centered invariance gives $\vec{\mathcal{X}}^{\boldsymbol{n}} = \text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$.*

Proof. Since TW$[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!]$ is the TW decomposition of tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, we have the k-contraction as follows,

$$\mathcal{X} = \mathcal{G}_1 \times_1^4 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C}. \tag{28}$$

Similarly, when $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is obtained by circularly reordering vector $(1, 2, \cdots, N)$, we firstly have

$$\text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!] = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{\boldsymbol{n}}, \tag{29}$$

which is of size $I_{n_1} \times I_{n_2} \times \cdots \times I_{n_N}$. Furthermore, the $(i_{n_1}, i_{n_2}, \cdots, i_{n_N})$-th element of TW$[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$ can be analytically expressed by

$$\begin{aligned}
\text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!](i_{n_1}, i_{n_2}, \cdots, i_{n_N}) &= (\mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{\boldsymbol{n}})(i_{n_1}, i_{n_2}, \cdots, i_{n_N}) \\
&= \sum_{l_{n_1}=1}^{L_{n_1}} \sum_{l_{n_2}=1}^{L_{n_2}} \cdots \sum_{l_{n_N}=1}^{L_{n_N}} \Big((\mathcal{G}_{n_1} \times_1^4 \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N})(i_{n_1}, \cdots, l_{n_N}) \vec{\mathcal{C}}^{\boldsymbol{n}}(l_{n_1}, l_{n_2}, \cdots, l_{n_N}) \Big) \\
&\;\;\vdots \\
&= \sum_{r_{n_1}=1}^{R_{n_1}} \cdots \sum_{r_{n_N}=1}^{R_{n_N}} \sum_{l_{n_1}=1}^{L_{n_1}} \sum_{l_{n_2}=1}^{L_{n_2}} \cdots \sum_{l_{n_N}=1}^{L_{n_N}} \Big(\mathcal{G}_{n_1}(r_{n_1}, i_{n_1}, l_{n_1}, r_{n_2}) \mathcal{G}_{n_2}(r_{n_2}, i_{n_2}, l_{n_2}, r_{n_3}) \\
&\qquad \cdots \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \cdots \\
&\qquad \mathcal{G}_{n_N}(r_{n_N}, i_{n_N}, l_{n_N}, r_{n_1}) \vec{\mathcal{C}}^{\boldsymbol{n}}(l_{n_1}, l_{n_2}, \cdots, l_{n_N}) \Big) \\
&= \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \\
&\qquad \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \\
&\qquad \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big) \\
&= \mathcal{X}(i_1, i_2, \cdots, i_N) \\
&= \vec{\mathcal{X}}^{\boldsymbol{n}}(i_{n_1}, i_{n_2}, \cdots, i_{n_N}),
\end{aligned} \tag{30}$$

where the fourth equation holds from the circularity of vector $\boldsymbol{n}$. Thus, $\vec{\mathcal{X}}^{\boldsymbol{n}} = \text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$ is clearly established. □

C. Proof of Theorem 2

Theorem B.2 (Core-Connected Invariance). *Assume that the generalized TW decomposition of $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is $\vec{\mathcal{X}}^{\boldsymbol{n}} = \text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$, where $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is any vector that circularly shifts vector $(1, 2, \cdots, N)$. Let vector $\boldsymbol{e} = (n_1, n_k, n_2, \cdots, n_{k-1}, n_{k+1}, \cdots, n_N)$ $(3 \le k < N,\ k \in \mathbb{Z})$, then*

$$\vec{\mathcal{X}}^{\boldsymbol{e}} = \mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \times_{2(N-k)+2,3,5,\cdots,2(N-k)+1,1}^{1,3,4,\cdots,N-k+2,N-k+4} \mathcal{G}_{n_{k+1},\cdots,n_N}, \tag{31}$$

where $\mathcal{G}_{n_2,\cdots,n_{k-1}} = \mathcal{G}_{n_2} \times_1^4 \cdots \times_1^{2(k-2)} \mathcal{G}_{n_{k-1}}$ and $\mathcal{G}_{n_{k+1},\cdots,n_N} = \mathcal{G}_{n_{k+1}} \times_1^4 \cdots \times_1^{2(N-k)} \mathcal{G}_{n_N}$.

Proof. From the perspective of tensor dimension, since $\vec{\mathcal{X}}^{\boldsymbol{n}} = \text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$, we easily have

$$\mathcal{G}_{n_2,\cdots,n_{k-1}} = \mathcal{G}_{n_2} \times_1^4 \cdots \times_1^{2(k-2)} \mathcal{G}_{n_{k-1}} \in \mathbb{R}^{R_{n_2} \times I_{n_2} \times L_{n_2} \times \cdots \times I_{n_{k-1}} \times L_{n_{k-1}} \times R_{n_k}}, \tag{32}$$

and

$$\mathcal{G}_{n_{k+1},\cdots,n_N} = \mathcal{G}_{n_{k+1}} \times_1^4 \cdots \times_1^{2(N-k)} \mathcal{G}_{n_N} \in \mathbb{R}^{R_{n_{k+1}} \times I_{n_{k+1}} \times L_{n_{k+1}} \times \cdots \times I_{n_N} \times L_{n_N} \times R_{n_1}}. \tag{33}$$

According to the pre-given vector $\boldsymbol{e} = (n_1, n_k, n_2, \cdots, n_{k-1}, n_{k+1}, \cdots, n_N)$ $(3 \le k < N,\ k \in \mathbb{Z})$, the corresponding tensor contractions yield

$$\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \in \mathbb{R}^{R_{n_1} \times I_{n_1} \times R_{n_2} \times L_{n_2} \times \cdots \times L_{n_{k-1}} \times L_{n_{k+1}} \times \cdots \times L_{n_N} \times R_{n_k} \times I_{n_k} \times R_{n_{k+1}}}. \tag{34}$$

Obviously, there are k common modes between $\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}$ and $\mathcal{G}_{n_2,\cdots,n_{k-1}}$, i.e., R_{n_2}, L_{n_2}, $\cdots$, $L_{n_{k-1}}$, and R_{n_k}. By eliminating the k modes, we can obtain

$$\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \in \mathbb{R}^{R_{n_1} \times I_{n_1} \times L_{n_{k+1}} \times \cdots \times L_{n_N} \times I_{n_k} \times R_{n_{k+1}} \times I_{n_2} \times \cdots \times I_{n_{k-1}}}. \tag{35}$$

Similarly, there are $N-k+2$ common modes between the above-generated tensor $\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}}$ and $\mathcal{G}_{n_{k+1},\cdots,n_N}$, i.e., R_{n_1}, $L_{n_{k+1}}$, $\cdots$, L_{n_N}, and $R_{n_{k+1}}$. Again, all contractions along these modes lead to

$$\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \times_{2(N-k)+2,3,5,\cdots,2(N-k)+1,1}^{1,3,4,\cdots,N-k+2,N-k+4} \mathcal{G}_{n_{k+1},\cdots,n_N}, \tag{36}$$

which is of size $I_{n_1} \times I_{n_k} \times I_{n_2} \times \cdots \times I_{n_{k-1}} \times I_{n_{k+1}} \times \cdots \times I_{n_N}$. Assume that $\mathcal{T} = \mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \times_{2(N-k)+2,3,5,\cdots,2(N-k)+1,1}^{1,3,4,\cdots,N-k+2,N-k+4} \mathcal{G}_{n_{k+1},\cdots,n_N}$, then the $(i_{n_1}, i_{n_k}, i_{n_2}, \cdots, i_{n_{k-1}}, i_{n_{k+1}}, \cdots, i_{n_N})$-th element of $\mathcal{T}$ can be correspondingly computed by

$$\begin{aligned}
\mathcal{T}(i_{n_1}, i_{n_k}, i_{n_2}, \cdots, i_{n_{k-1}}, i_{n_{k+1}}, \cdots, i_{n_N}) &= \sum_{r_{n_1}=1}^{R_{n_1}} \cdots \sum_{r_{n_N}=1}^{R_{n_N}} \sum_{l_{n_1}=1}^{L_{n_1}} \cdots \sum_{l_{n_N}=1}^{L_{n_N}} \Big(\mathcal{G}_{n_1}(r_{n_1}, i_{n_1}, l_{n_1}, r_{n_2}) \mathcal{G}_{n_2}(r_{n_2}, i_{n_2}, l_{n_2}, r_{n_3}) \\
&\qquad \cdots \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \cdots \\
&\qquad \mathcal{G}_{n_N}(r_{n_N}, i_{n_N}, l_{n_N}, r_{n_1}) \vec{\mathcal{C}}^{\boldsymbol{e}}(l_{n_1}, l_{n_k}, \cdots, l_{n_N}) \Big) \\
&= \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_N=1}^{R_N} \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \cdots \sum_{l_N=1}^{L_N} \Big(\mathcal{G}_1(r_1, i_1, l_1, r_2) \mathcal{G}_2(r_2, i_2, l_2, r_3) \\
&\qquad \cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1}) \cdots \\
&\qquad \mathcal{G}_N(r_N, i_N, l_N, r_1) \mathcal{C}(l_1, l_2, \cdots, l_N) \Big) \\
&= \mathcal{X}(i_1, i_2, \cdots, i_N) \\
&= \vec{\mathcal{X}}^{\boldsymbol{e}}(i_{n_1}, i_{n_k}, i_{n_2}, \cdots, i_{n_{k-1}}, i_{n_{k+1}}, \cdots, i_{n_N}),
\end{aligned} \tag{37}$$

which is definitely equivalent to $\mathcal{T} = \vec{\mathcal{X}}^{\boldsymbol{e}}$, i.e.,

$$\vec{\mathcal{X}}^{\boldsymbol{e}} = \mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \times_{2(N-k)+2,3,5,\cdots,2(N-k)+1,1}^{1,3,4,\cdots,N-k+2,N-k+4} \mathcal{G}_{n_{k+1},\cdots,n_N}. \tag{38}$$

The proof is completed. □

D. Proof of Theorem 3

Theorem B.3 (**Tensor Subwheel Equation**). *Assume that the generalized TW decomposition of $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is $\vec{\mathcal{X}}^{\boldsymbol{n}} = TW[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$, where $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is any vector that circularly shifts vector $(1, 2, \cdots, N)$. Let vectors $\boldsymbol{u} = (N+1, N+2, 1, 2, \cdots, N)$ and $\boldsymbol{v} = (2, 4, \cdots, 2N, 1, 3, \cdots, 2N-1)$, then there inherently exists the following two tensor subwheel equations,*

$$\mathbf{X}_{<n_N>} = (\mathbf{G}_{n_N})_{(2)} (\mathbf{M}_{\neq n_N})_{[\boldsymbol{u};3]} \tag{39}$$

and

$$\vec{\mathbf{x}}^{\boldsymbol{n}}_{[(1:N);0]} = \vec{\mathbf{c}}^{\boldsymbol{n}}_{[(1:N);0]} (\mathbf{N}_{\neq \mathcal{C}})_{[\boldsymbol{v};N]}, \tag{40}$$

where $\mathcal{M}_{\neq n_N} \in \mathbb{R}^{R_{n_1} \times I_{n_1} \times \cdots \times I_{n_{N-1}} \times R_{n_N} \times L_{n_N}}$ is an $(N+2)$th-order subwheel tensor, which merges all TW factors but $\mathcal{G}_{n_N}$, i.e., $\mathcal{M}_{\neq n_N} = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_1^{2N-2} \mathcal{G}_{n_{N-1}} \times_{1,2,\cdots,N-1}^{3,5,\cdots,2N-1} \vec{\mathcal{C}}^{\boldsymbol{n}}$, and $\mathcal{N}_{\neq \mathcal{C}} \in \mathbb{R}^{I_{n_1} \times L_{n_1} \times \cdots \times I_{n_N} \times L_{n_N}}$ is another $2N$th-order subwheel tensor obtained by only merging $\{\mathcal{G}_{n_k}\}_{k=1}^N$, i.e., $\mathcal{N}_{\neq \mathcal{C}} = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N}$.

Proof. Since $\vec{\mathcal{X}}^{\boldsymbol{n}} = \text{TW}[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$ is the TW decomposition of $\vec{\mathcal{X}}^{\boldsymbol{n}} \in \mathbb{R}^{I_{n_1} \times I_{n_2} \times \cdots \times I_{n_N}}$, the concrete contraction-based expression can be given by

$$\vec{\mathcal{X}}^{\boldsymbol{n}} = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{\boldsymbol{n}}. \tag{41}$$

Following Theorem B.1, we rewrite its form of tensor k-contraction as follows,

$$\vec{\mathcal{X}}^{(n_N, n_1, n_2, \cdots, n_{N-1})} = \mathcal{G}_{n_N} \times_1^4 \mathcal{G}_{n_1} \times_1^6 \cdots \times_1^{2k+2} \mathcal{G}_{n_k} \times_1^{2k+4} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_{N-1}} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{(n_N, n_1, n_2, \cdots, n_{N-1})}, \tag{42}$$

which amounts to

$$\vec{\mathcal{X}}^{(n_N, n_1, n_2, \cdots, n_{N-1})} = \mathcal{G}_{n_N} \times_1^4 \mathcal{G}_{n_1} \times_1^6 \cdots \times_1^{2k+2} \mathcal{G}_{n_k} \times_1^{2k+4} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_{N-1}} \times_{N,1,2,\cdots,N-1}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{\boldsymbol{n}}. \tag{43}$$

Relying upon Property A.3, the relationship in (43) is further written as

$$\vec{\mathcal{X}}^{(n_N,n_1,n_2,\cdots,n_{N-1})} = \mathcal{G}_{n_N} \times_{N+1,N+2,1}^{1,3,4} (\mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_1^{2N-2} \mathcal{G}_{n_{N-1}} \times_{1,2,\cdots,N-1}^{3,5,\cdots,2N-1} \vec{\mathcal{C}}^{\boldsymbol{n}}). \tag{44}$$

According to Definition A.2, formula (44) can further be converted to the corresponding matrix product, i.e.,

$$\mathbf{X}_{<n_N>} = (\mathbf{G}_{n_N})_{(2)}(\mathbf{M}_{\neq n_N})_{[\boldsymbol{u};3]}, \tag{45}$$

where $\mathcal{M}_{\neq n_N} = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_1^{2N-2} \mathcal{G}_{n_{N-1}} \times_{1,2,\cdots,N-1}^{3,5,\cdots,2N-1} \vec{\mathcal{C}}^{\boldsymbol{n}}$ and $\boldsymbol{u} = (N+1, N+2, 1, 2, \cdots, N)$ are required. For another, the relationship in (41) is equivalent to

$$\vec{\mathcal{X}}^{\boldsymbol{n}} = (\mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N}) \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \vec{\mathcal{C}}^{\boldsymbol{n}}. \tag{46}$$

Relying upon Property A.2, the above formula (46) directly allows

$$\vec{\mathcal{X}}^{\boldsymbol{n}} = \vec{\mathcal{C}}^{\boldsymbol{n}} \times_{2,4,\cdots,2N}^{1,2,\cdots,N} (\mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N}), \tag{47}$$

whose corresponding matrix product can be similarly given as

$$\vec{\mathbf{x}}^{\boldsymbol{n}}_{[(1:N);0]} = \vec{\mathbf{c}}^{\boldsymbol{n}}_{[(1:N);0]}(\mathbf{N}_{\neq \mathcal{C}})_{[\boldsymbol{v};N]}, \tag{48}$$

where $\mathcal{N}_{\neq \mathcal{C}} = \mathcal{G}_{n_1} \times_1^4 \cdots \times_1^{2k} \mathcal{G}_{n_k} \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_{n_N}$ and $\boldsymbol{v} = (2, 4, \cdots, 2N, 1, 3, \cdots, 2N-1)$ are also accordingly required. This completes the proof. □

E. Proof of Theorem 4

Theorem B.4. *Assume that $\mathcal{X} = TW[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!]$ with N ring factors $\mathcal{G}_k \in \mathbb{R}^{R_k \times I_k \times L_k \times R_{k+1}}$, then*

$$\mathtt{Rank}(\mathbf{X}_{(k)}) = \mathtt{Rank}(\mathbf{X}_{<k>}) \leq L_k \prod_{i=k}^{k+1} R_i, \quad k = 1, 2, \cdots, N. \tag{49}$$

Proof. According to Theorem B.3, we have the subwheel equations for all $k = 1, 2, \cdots, N$, as follows,

$$\mathbf{X}_{<k>} = (\mathbf{G}_k)_{(2)}(\mathbf{M}_{\neq k})_{[\boldsymbol{u};3]}, \tag{50}$$

where $\boldsymbol{u} = (N+1, N+2, 1, 2, \cdots, N)$. Then, we can justify the following inequality,

$$\begin{aligned} \mathtt{Rank}(\mathbf{X}_{(k)}) = \mathtt{Rank}(\mathbf{X}_{<k>}) &\leq \mathtt{Rank}((\mathbf{G}_k)_{(2)}) \wedge \mathtt{Rank}((\mathbf{M}_{\neq k})_{[\boldsymbol{u};3]}) \\ &\leq \mathtt{Rank}((\mathbf{G}_k)_{(2)}) \\ &\leq I_k \wedge R_k L_k R_{k+1} \\ &\leq L_k \prod_{i=k}^{k+1} R_i. \end{aligned} \tag{51}$$

The proof is completed. □

Appendix C
Proof of Proposition 1

Before proving Proposition C.1, we need to provide a corollary of Theorem B.2. Such a corollary also explores a type of generalized TW decomposition, in which the ring factor $\mathcal{G}_k \in \mathbb{R}^{R_k \times I_k \times L_k \times R_{k+1}}$ for $\forall k = 1, 2, \cdots, N$, can preferentially contract with the core factor $\mathcal{C} \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_k \times \cdots \times L_N}$ without involving the tensor permutation.

A. Corollary of Theorem 2

Corollary C.1. *Assume that the generalized TW decomposition of* $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ *is* $\vec{\mathcal{X}}^{\boldsymbol{n}} = TW[\![\{\mathcal{G}_{n_k}\}_{k=1}^N; \vec{\mathcal{C}}^{\boldsymbol{n}}]\!]$, *where* $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ *is any vector that circularly shifts vector* $(1, 2, \cdots, N)$. *Let vectors* $\boldsymbol{e} = (n_1, n_k, n_2, \cdots, n_{k-1}, n_{k+1}, \cdots, n_N)$ $(3 \le k < N,\ k \in \mathbb{Z})$ *and* $\boldsymbol{u} = (1, 2, 3, n_2+2, n_3+2, \cdots, n_{N-n_1+1}+2, n_{N-n_1+2}+3, \cdots, n_{N-1}+3, n_N+3)$, *then*

$$\vec{\mathcal{X}}^{\boldsymbol{e}} = \overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k} \times_{1,3,\cdots,2k-3,2k-2}^{3,4,\cdots,k+1,N+2} \mathcal{G}_{n_2,\cdots,n_{k-1}} \times_{2(N-k)+2,3,5,\cdots,2(N-k)+1,1}^{1,3,4,\cdots,N-k+2,N-k+4} \mathcal{G}_{n_{k+1},\cdots,n_N}, \tag{52}$$

where $\mathcal{G}_{n_2,\cdots,n_{k-1}} = \mathcal{G}_{n_2} \times_1^4 \cdots \times_1^{2(k-2)} \mathcal{G}_{n_{k-1}}$ *and* $\mathcal{G}_{n_{k+1},\cdots,n_N} = \mathcal{G}_{n_{k+1}} \times_1^4 \cdots \times_1^{2(N-k)} \mathcal{G}_{n_N}$.

Proof. Compared with Theorem B.2, Corollary C.1 mainly replaces $\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}$ with $\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k}$. Accordingly, the proof of Corollary C.1 can be simplified to prove

$$\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k} = \mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}. \tag{53}$$

Firstly, the corresponding tensor contractions enable $\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k}$ to be of size $R_{n_1} \times I_{n_1} \times R_{n_2} \times L_{n_2} \times \cdots \times L_{n_{k-1}} \times L_{n_{k+1}} \times \cdots \times L_{n_N} \times R_{n_k} \times I_{n_k} \times R_{n_{k+1}}$, which is consistent with that of $\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}$. Furthermore, the $(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}, r_{n_k}, i_{n_k}, r_{n_{k+1}})$-th element of $\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k}$ can be formulated by

$$\begin{aligned}
&\left(\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k}\right)(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}, r_{n_k}, i_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} \left(\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}}\right)(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_2}, \cdots, l_{n_k}, \cdots, l_{n_N}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} (\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_{N-n_1+2}}, \cdots, l_{n_N}, l_{n_2}, \cdots, l_{n_{N-n_1+1}}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} (\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})(r_{n_1}, i_{n_1}, r_{n_2}, \underbrace{l_1, \cdots,}_{n_1-1} \underbrace{\cdots, l_N}_{N-n_1}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} \sum_{l_{n_1}}^{L_{n_1}} \mathcal{G}_{n_1}(r_{n_1}, i_{n_1}, l_{n_1}, r_{n_2}) \mathcal{C}(\underbrace{l_1, \cdots,}_{n_1-1} l_{n_1}, \underbrace{\cdots, l_N}_{N-n_1}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}).
\end{aligned} \tag{54}$$

Likewisely, the $(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}, r_{n_k}, i_{n_k}, r_{n_{k+1}})$-th element of $\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}$ can be given by

$$\begin{aligned}
&(\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k})(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}, r_{n_k}, i_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} (\mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}})(r_{n_1}, i_{n_1}, r_{n_2}, l_{n_k}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} \sum_{l_{n_1}}^{L_{n_1}} \mathcal{G}_{n_1}(r_{n_1}, i_{n_1}, l_{n_1}, r_{n_2}) \vec{\mathcal{C}}^{\boldsymbol{e}}(l_{n_1}, l_{n_k}, l_{n_2}, \cdots, l_{n_{k-1}}, l_{n_{k+1}}, \cdots, l_{n_N}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}) \\
&\quad = \sum_{l_{n_k}}^{L_{n_k}} \sum_{l_{n_1}}^{L_{n_1}} \mathcal{G}_{n_1}(r_{n_1}, i_{n_1}, l_{n_1}, r_{n_2}) \mathcal{C}(\underbrace{l_1, \cdots,}_{n_1-1} l_{n_1}, \underbrace{\cdots, l_N}_{N-n_1}) \mathcal{G}_{n_k}(r_{n_k}, i_{n_k}, l_{n_k}, r_{n_{k+1}}).
\end{aligned} \tag{55}$$

From formulas (54)-(55), there obviously exists $\overrightarrow{(\mathcal{G}_{n_1} \times_{n_1}^{3} \mathcal{C})}^{\boldsymbol{u}} \times_3^{k+2} \mathcal{G}_{n_k} = \mathcal{G}_{n_1} \times_1^3 \vec{\mathcal{C}}^{\boldsymbol{e}} \times_3^4 \mathcal{G}_{n_k}$. The proof is completed. □

B. Proof of Proposition 1

Proposition C.1. *Let TW*$[\![\{\mathcal{G}_k\}_{k=1}^N;\mathcal{C}]\!]$ *be the TW decomposition of tensor* $\mathcal{X}\in\mathbb{R}^{I_1\times I_2\times\cdots\times I_N}$, *where* $\mathcal{G}_k\in\mathbb{R}^{R_k\times I_k\times L_k\times R_{k+1}}$, $k=1,2,\cdots,N$, *and* $\mathcal{C}\in\mathbb{R}^{L_1\times L_2\times\cdots\times L_k\times\cdots\times L_N}$. *Assume that for* $\forall k$, $\bar{R}_k=R_k+p_k$ *and* $\bar{L}_k=L_k+q_k$, *where* $p_k,q_k\in\mathbb{N}$, *and denote the factors* $\mathcal{G}_k$, $\mathcal{C}$, $\bar{\mathcal{G}}_k$, *and* $\bar{\mathcal{C}}$ *by* $\bar{\mathcal{G}}_k^{(0)}\in\mathbb{R}^{R_k\times I_k\times L_k\times R_{k+1}}$, $\bar{\mathcal{C}}^{(0)}\in\mathbb{R}^{L_1\times L_2\times\cdots\times L_k\times\cdots\times L_N}$, $\bar{\mathcal{G}}_k^{(4)}\in\mathbb{R}^{\bar{R}_k\times I_k\times\bar{L}_k\times\bar{R}_{k+1}}$, *and* $\bar{\mathcal{C}}^{(N)}\in\mathbb{R}^{\bar{L}_1\times\bar{L}_2\times\cdots\times\bar{L}_k\times\cdots\times\bar{L}_N}$. *When* $\bar{\mathcal{G}}_k^{(i)}$, $i=1,2,3,4$, *and* $\bar{\mathcal{C}}^{(j)}$, $j=1,2,\cdots,N$, *are recursively defined by*

$$\left(\bar{\mathbf{G}}_k^{(i)}\right)_{(i)}=\begin{bmatrix}\left(\bar{\mathbf{G}}_k^{(i-1)}\right)_{(i)}\\ \mathbf{0}\end{bmatrix},\ k=1,2,\cdots,N, \tag{56}$$

and

$$\left(\bar{\mathbf{C}}^{(j)}\right)_{(j)}=\begin{bmatrix}\left(\bar{\mathbf{C}}^{(j-1)}\right)_{(j)}\\ \mathbf{0}\end{bmatrix}, \tag{57}$$

where $\mathbf{0}$ *implies the all-zeros matrix with appropriate size, the factors* $\bar{\mathcal{G}}_k$, $k=1,2,\cdots,N$, *and* $\bar{\mathcal{C}}$ *satisfy* $\mathcal{X}=$ *TW*$[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N;\bar{\mathcal{C}}]\!]$.

Proof. To prove Proposition C.1, we also let $\hat{\mathcal{G}}_k^{(0)}=\bar{\mathcal{G}}_k^{(0)}=\mathcal{G}_k\in\mathbb{R}^{R_k\times I_k\times L_k\times R_{k+1}}$, and define $\hat{\mathcal{G}}_k^{(i)}$, $i=1,2,3,4$, as follows,

$$\left(\hat{\mathbf{G}}_k^{(i)}\right)_{(5-i)}=\begin{bmatrix}\left(\hat{\mathbf{G}}_k^{(i-1)}\right)_{(5-i)}\\ \mathbf{0}\end{bmatrix},\ k=1,2,\cdots,N. \tag{58}$$

Therefore, $\hat{\mathcal{G}}_k^{(4)}=\bar{\mathcal{G}}_k^{(4)}=\bar{\mathcal{G}}_k\in\mathbb{R}^{\bar{R}_k\times I_k\times\bar{L}_k\times\bar{R}_{k+1}}$. Besides, we individually define $\tilde{\mathcal{G}}_k\in\mathbb{R}^{R_k\times I_k\times\bar{L}_k\times R_{k+1}}$ by

$$\left(\tilde{\mathbf{G}}_k\right)_{(3)}=\begin{bmatrix}\left(\mathbf{G}_k\right)_{(3)}\\ \mathbf{0}\end{bmatrix},\ k=1,2,\cdots,N. \tag{59}$$

According to the contraction principle of TW decomposition, we naturally have

$$\text{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N;\bar{\mathcal{C}}]\!]=\bar{\mathcal{G}}_1\times_1^4\cdots\times_1^{2k}\bar{\mathcal{G}}_k\times_1^{2k+2}\cdots\times_{1,4}^{2N,1}\bar{\mathcal{G}}_N\times_{1,2,\cdots,N}^{2,4,\cdots,2N}\bar{\mathcal{C}}, \tag{60}$$

where factors $\bar{\mathcal{G}}_k\in\mathbb{R}^{\bar{R}_k\times I_k\times\bar{L}_k\times\bar{R}_{k+1}}$, $k=1,2,\cdots,N$, and $\bar{\mathcal{C}}\in\mathbb{R}^{\bar{L}_1\times\bar{L}_2\times\cdots\times\bar{L}_k\times\cdots\times\bar{L}_N}$ are defined by formulas (56)-(57). Following Definition A.2, we firstly give

$$\begin{aligned}\bar{\mathcal{G}}_1\times_1^4\bar{\mathcal{G}}_2=\bar{\mathcal{G}}_1^{(4)}\times_1^4\hat{\mathcal{G}}_2^{(4)}&=\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\bar{\mathbf{G}}_1^{(4)}\right)^\top_{[(4,1,2,3);1]}\left(\hat{\mathbf{G}}_2^{(4)}\right)_{[(1,2,3,4);1]}\right)\\&=\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\begin{bmatrix}\left(\bar{\mathbf{G}}_1^{(3)}\right)_{(4)}\\ \mathbf{0}\end{bmatrix}^\top\begin{bmatrix}\left(\hat{\mathbf{G}}_2^{(3)}\right)_{(1)}\\ \mathbf{0}\end{bmatrix}\right)\\&=\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\bar{\mathbf{G}}_1^{(3)}\right)^\top_{(4)}\left(\hat{\mathbf{G}}_2^{(3)}\right)_{(1)}+\mathbf{0}^\top\mathbf{0}\right)\\&=\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\bar{\mathbf{G}}_1^{(3)}\right)^\top_{[(4,1,2,3);1]}\left(\hat{\mathbf{G}}_2^{(3)}\right)_{[(1,2,3,4);1]}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\hat{\mathcal{G}}_2^{(3)}.\end{aligned} \tag{61}$$

By combining the above formula (61) and transforming the contraction form, we further have

$$\begin{aligned}\bar{\mathcal{G}}_1\times_1^4\bar{\mathcal{G}}_2\times_1^6\bar{\mathcal{G}}_3=\bar{\mathcal{G}}_1^{(3)}\times_1^4\hat{\mathcal{G}}_2^{(3)}\times_1^6\bar{\mathcal{G}}_3&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\left(\hat{\mathcal{G}}_2^{(3)}\times_1^4\hat{\mathcal{G}}_3^{(4)}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\hat{\mathbf{G}}_2^{(3)}\right)^\top_{[(4,1,2,3);1]}\left(\hat{\mathbf{G}}_3^{(4)}\right)_{[(1,2,3,4);1]}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\hat{\mathbf{G}}_2^{(2)}\right)^\top_{[(4,1,2,3);1]}\left(\hat{\mathbf{G}}_3^{(4)}\right)_{[(1,2,3,4);1]}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\begin{bmatrix}\left(\tilde{\mathbf{G}}_2\right)_{(4)}\\ \mathbf{0}\end{bmatrix}^\top\begin{bmatrix}\left(\hat{\mathbf{G}}_3^{(3)}\right)_{(1)}\\ \mathbf{0}\end{bmatrix}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\tilde{\mathbf{G}}_2\right)^\top_{(4)}\left(\hat{\mathbf{G}}_3^{(3)}\right)_{(1)}+\mathbf{0}^\top\mathbf{0}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\texttt{Fold}_{[(1,2,\cdots,6);3]}\left(\left(\tilde{\mathbf{G}}_2\right)^\top_{[(4,1,2,3);1]}\left(\hat{\mathbf{G}}_3^{(3)}\right)_{[(1,2,3,4);1]}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\left(\tilde{\mathcal{G}}_2\times_1^4\hat{\mathcal{G}}_3^{(3)}\right)\\&=\bar{\mathcal{G}}_1^{(3)}\times_1^4\tilde{\mathcal{G}}_2\times_1^6\hat{\mathcal{G}}_3^{(3)},\end{aligned} \tag{62}$$

where $\left(\hat{\mathbf{G}}_2^{(3)}\right)^{\top}_{[(4,1,2,3);1]} = \left(\hat{\mathbf{G}}_2^{(2)}\right)^{\top}_{[(4,1,2,3);1]}$ holds since the I_k, $k = 1, 2, \cdots, N$, mode is maintained without array-padding from formula (58). Similar to formula (62), we can directly induce

$$\mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!] = \bar{\mathcal{G}}_1^{(3)} \times_1^4 \tilde{\mathcal{G}}_2 \times_1^6 \cdots \times_1^{2k} \tilde{\mathcal{G}}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \tilde{\mathcal{G}}_{N-1} \times_{1,4}^{2N,1} \hat{\mathcal{G}}_N^{(3)} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \bar{\mathcal{C}}. \tag{63}$$

Let vector $\boldsymbol{u} = (N, 1, 2, \cdots, N-1)$. Then Theorem B.1 can rewrite equation (63) as

$$\overrightarrow{\left(\mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!]\right)}^{\boldsymbol{u}} = \hat{\mathcal{G}}_N^{(3)} \times_1^4 \bar{\mathcal{G}}_1^{(3)} \times_1^6 \tilde{\mathcal{G}}_2 \times_1^8 \cdots \times_1^{2k+2} \tilde{\mathcal{G}}_k \times_1^{2k+4} \cdots \times_{1,4}^{2N,1} \tilde{\mathcal{G}}_{N-1} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \overrightarrow{(\bar{\mathcal{C}})}^{\boldsymbol{u}}, \tag{64}$$

which is equivalent to

$$\overrightarrow{\left(\mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!]\right)}^{\boldsymbol{u}} = \tilde{\mathcal{G}}_N \times_1^4 \tilde{\mathcal{G}}_1 \times_1^6 \tilde{\mathcal{G}}_2 \times_1^8 \cdots \times_1^{2k+2} \tilde{\mathcal{G}}_k \times_1^{2k+4} \cdots \times_{1,4}^{2N,1} \tilde{\mathcal{G}}_{N-1} \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \overrightarrow{(\bar{\mathcal{C}})}^{\boldsymbol{u}}. \tag{65}$$

Subsequently, we obtain

$$\mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!] = \tilde{\mathcal{G}}_1 \times_1^4 \tilde{\mathcal{G}}_2 \times_1^6 \cdots \times_1^{2k} \tilde{\mathcal{G}}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \tilde{\mathcal{G}}_{N-1} \times_{1,4}^{2N,1} \tilde{\mathcal{G}}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \bar{\mathcal{C}}. \tag{66}$$

By using a similar manner (i.e., adjusting the priority of tensor contraction), Corollary C.1 can establish

$$\begin{aligned}
\mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!] &= \tilde{\mathcal{G}}_1 \times_1^4 \tilde{\mathcal{G}}_2 \times_1^6 \cdots \times_1^{2k} \tilde{\mathcal{G}}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \tilde{\mathcal{G}}_{N-1} \times_{1,4}^{2N,1} \tilde{\mathcal{G}}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \bar{\mathcal{C}} \\
&= \tilde{\mathcal{G}}_1 \times_1^4 \tilde{\mathcal{G}}_2 \times_1^6 \cdots \times_1^{2k} \tilde{\mathcal{G}}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \tilde{\mathcal{G}}_{N-1} \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \bar{\mathcal{C}}^{(N-1)} \\
&\ \ \vdots \\
&= \tilde{\mathcal{G}}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \mathcal{G}_{N-1} \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \bar{\mathcal{C}}^{(1)} \\
&= \mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_1^{2N-2} \mathcal{G}_{N-1} \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C} \\
&= \mathrm{TW}[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!] \\
&= \mathcal{X}.
\end{aligned} \tag{67}$$

Accordingly, $\mathcal{X} = \mathrm{TW}[\![\{\bar{\mathcal{G}}_k\}_{k=1}^N; \bar{\mathcal{C}}]\!]$ holds. The proof is completed. □

Appendix D
Proof of Theorem 5

A. Main Preliminaries

Given the unfolding matrix $\mathbf{X}_{<1>} \in \mathbb{R}^{I_1 \times \prod_{i=2}^{N} I_i}$, let the compact SVD of $\mathbf{X}_{<1>}$ be

$$\mathbf{X}_{<1>} = \begin{bmatrix} \mathbf{U} & \mathbf{U}_\epsilon \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma} & \\ & \mathbf{\Sigma}_\epsilon \end{bmatrix} \begin{bmatrix} \mathbf{V}^\top \\ \mathbf{V}_\epsilon^\top \end{bmatrix} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top + \mathbf{U}_\epsilon \mathbf{\Sigma}_\epsilon \mathbf{V}_\epsilon^\top, \tag{68}$$

where $\mathbf{U} \in \mathbb{R}^{I_1 \times R_1 L_1 R_2}$ includes the top $R_1 L_1 R_2$ left singular vectors, and $\mathbf{U}_\epsilon \mathbf{\Sigma}_\epsilon \mathbf{V}_\epsilon^\top$ implies the possible truncation error. From the properties of the SVD, there exists

$$\mathbf{U}^\top \mathbf{U}_\epsilon \mathbf{\Sigma}_\epsilon \mathbf{V}_\epsilon^\top = \mathbf{0}, \quad \text{and} \quad \mathbf{\Sigma}\mathbf{V}^\top = \mathbf{U}^\top \mathbf{X}_{<1>}. \tag{69}$$

Besides, the truncation error can be expressed as

$$\left\| \mathbf{X}_{<1>} - \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \right\|_F = \left\| \mathbf{U}_\epsilon \mathbf{\Sigma}_\epsilon \mathbf{V}_\epsilon^\top \right\|_F = \left\| \mathbf{\Sigma}_\epsilon \right\|_F = \sqrt{\sum_{i=R_1 L_1 R_2 + 1}^{I_1 \wedge \prod_{i=2}^{N} I_i} \sigma_i^2 \left(\mathbf{X}_{<1>} \right)}, \tag{70}$$

where $\sigma_i(\cdot)$ represents the i-th singular value of matrix $\mathbf{X}_{<1>}$.

B. Proof of Theorem 5

Theorem D.1 (Approximation Error). *Given an Nth-order tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ and its predefined TW-ranks. Assume that the factors $\mathcal{G}_k \in \mathbb{R}^{R_k \times I_k \times L_k \times R_{k+1}}$, $k = 1, 2, \cdots, N$, and $\mathcal{C} \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_k \times \cdots \times L_N}$ are generated by Algorithm 1 of the main text. Then, the TW-format tensor, i.e., TW$[\![\{\mathcal{G}_k\}_{k=1}^{N}; \mathcal{C}]\!]$, satisfies*

$$\begin{aligned} \left\| \mathcal{X} - \mathit{TW}[\![\{\mathcal{G}_k\}_{k=1}^{N}; \mathcal{C}]\!] \right\|_F \leq & \sqrt{\sum_{i=R_1 L_1 R_2 + 1}^{I_1 \wedge \prod_{i=2}^{N} I_i} \sigma_i^2 \left(\mathbf{X}_{<1>} \right)} + \sum_{k=2}^{N-1} \sqrt{\sum_{i=L_k R_{k+1}+1}^{R_k I_k \wedge \prod_{i=k+1}^{N} I_i R_1 \prod_{j=1}^{k-1} L_j} \sigma_i^2 \left(\left(\mathbf{M}_{\neq(1:k-1)} \right)_{[(1:N+2);2]} \right)} \\ & + \sqrt{\sum_{i=L_N+1}^{R_N I_N R_1 \wedge \prod_{j=1}^{N-1} L_j} \sigma_i^2 \left(\left(\mathbf{M}_{\neq(1:N-1)} \right)_{[(1:N+2);3]} \right)}, \end{aligned} \tag{71}$$

where $\mathcal{M}_{\neq 1} = \mathtt{Fold}_{[(N+1,N+2,1:N);3]}((\mathbf{G}_1)_{(2)}^\top \mathbf{X}_{<1>})$, $\mathcal{M}_{\neq(1:k)} = \mathtt{Fold}_{[z;2]}((\mathbf{G}_k)_{[(1:4);2]}^\top \left(\mathbf{M}_{\neq(1:k-1)} \right)_{[(1:N+2);2]})$, $k = 2, 3, \cdots, N-1$, and $\sigma_i(\cdot)$ indicates the i-th singular value.

Proof. According to Algorithm 1 of the main text, the approximation error can be firstly computed as

$$\begin{aligned} & \left\| \mathcal{X} - \mathrm{TW}[\![\{\mathcal{G}_k\}_{k=1}^{N}; \mathcal{C}]\!] \right\|_F = \left\| \mathbf{X}_{<1>} - (\mathbf{G}_1)_{(2)} \overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}} \right\|_F \\ & = \left\| \mathbf{X}_{<1>} - (\mathbf{G}_1)_{(2)} \left(\overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}} + (\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} - (\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} \right) \right\|_F \\ & \leq \left\| \mathbf{X}_{<1>} - (\mathbf{G}_1)_{(2)} (\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} \right\|_F + \left\| (\mathbf{G}_1)_{(2)} \left((\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} - \overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}} \right) \right\|_F \\ & = \sqrt{\sum_{i=R_1 L_1 R_2 + 1}^{I_1 \wedge \prod_{i=2}^{N} I_i} \sigma_i^2 \left(\mathbf{X}_{<1>} \right)} + \left\| (\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} - \overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}} \right\|_F, \end{aligned} \tag{72}$$

where $(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}$ is given by $(\mathbf{G}_1)_{(2)}^{\top}\mathbf{X}_{<1>}$, and $\overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}}$ denotes the approximation of $(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}$ with $N-1$ truncation errors. Furthermore, we easily have

$$\begin{aligned}
&\left\|(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]} - \overline{(\mathbf{M}_{\neq 1})_{[(N+1,N+2,1:N);3]}}\right\|_F \\
&= \left\|(\mathbf{M}_{\neq 1})_{[(1:N+2);2]} - \overline{(\mathbf{M}_{\neq 1})_{[(1:N+2);2]}}\right\|_F \\
&= \left\|(\mathbf{M}_{\neq 1})_{[(1:N+2);2]} - (\mathbf{G}_2)_{[(1:4);2]}\overline{\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}}\right\|_F \\
&= \left\|(\mathbf{M}_{\neq 1})_{[(1:N+2);2]} - (\mathbf{G}_2)_{[(1:4);2]}\left(\overline{\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}} + \left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]} - \left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}\right)\right\|_F \\
&\leq \left\|(\mathbf{M}_{\neq 1})_{[(1:N+2);2]} - (\mathbf{G}_2)_{[(1:4);2]}\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}\right\|_F + \left\|(\mathbf{G}_2)_{[(1:4);2]}\left(\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]} - \overline{\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}}\right)\right\|_F \\
&= \sqrt{\sum_{i=L_2R_3+1}^{R_2I_2\wedge\prod_{i=3}^{N}I_iR_1L_1} \sigma_i^2\left((\mathbf{M}_{\neq 1})_{[(1:N+2);2]}\right)} + \left\|\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]} - \overline{\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}}\right\|_F,
\end{aligned} \tag{73}$$

where $\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}$ is given by $(\mathbf{G}_2)_{[(1:4);2]}^{\top}(\mathbf{M}_{\neq 1})_{[(1:N+2);2]}$, and $\overline{\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}}$ implies the approximation of $\left(\mathbf{M}_{\neq(1:2)}\right)_{[z;2]}$ with $N-2$ truncation errors. Proceeding by induction, we certainly have

$$\begin{aligned}
\left\|\mathcal{X} - \mathrm{TW}[\![\{\mathcal{G}_k\}_{k=1}^{N};\mathcal{C}]\!]\right\|_F \leq & \sqrt{\sum_{i=R_1L_1R_2+1}^{I_1\wedge\prod_{i=2}^{N}I_i} \sigma_i^2\left(\mathbf{X}_{<1>}\right)} + \sum_{k=2}^{N-1}\sqrt{\sum_{i=L_kR_{k+1}+1}^{R_kI_k\wedge\prod_{i=k+1}^{N}I_iR_1\prod_{j=1}^{k-1}L_j} \sigma_i^2\left(\left(\mathbf{M}_{\neq(1:k-1)}\right)_{[(1:N+2);2]}\right)} \\
& + \sqrt{\sum_{i=L_N+1}^{R_NI_NR_1\wedge\prod_{j=1}^{N-1}L_j} \sigma_i^2\left(\left(\mathbf{M}_{\neq(1:N-1)}\right)_{[(1:N+2);3]}\right)},
\end{aligned} \tag{74}$$

where $\mathcal{M}_{\neq 1} = \mathtt{Fold}_{[(N+1,N+2,1:N);3]}((\mathbf{G}_1)_{(2)}^{\top}\mathbf{X}_{<1>})$ and $\mathcal{M}_{\neq(1:k)} = \mathtt{Fold}_{[z;2]}((\mathbf{G}_k)_{[(1:4);2]}^{\top}\left(\mathbf{M}_{\neq(1:k-1)}\right)_{[(1:N+2);2]})$, $k = 2, 3, \cdots, N-1$. The proof is then completed. $\square$

APPENDIX E
PROOF OF THEOREM 6

A. Proof of Theorem 6

Theorem E.1 (Local Convergence). *Let $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ be the sequence generated by Algorithm 3 of the main text, then it globally converges to a critical point (i.e., local minimum point) of $\Phi(\mathcal{Z})$.*

Proof. According to the finite length property (see [2, Theorem 1]), the sequence $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ can converge to the critical point of $\Phi(\mathcal{Z})$, since Lemmas E.1-E.4 are satisfied (see Appendix E-B). □

B. Proofs of lemmas 1-4

Before proving the following Lemmas E.1-E.4, we firstly rewrite the proposed TW-TC model as follows,

$$\min_{\mathcal{Z}} \ \Phi(\mathcal{Z}) = h(\mathcal{Z}) + f(\mathcal{X}), \tag{75}$$

where $\mathcal{Z} = (\mathcal{X}, \mathcal{G}_{1:N}, \mathcal{C})$, $h(\mathcal{Z}) = 1/2\|\mathcal{X} - \text{TW}[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!]\|_F^2$, and $f(\mathcal{X}) = \iota(\mathcal{X})$.

Lemma E.1. *The $\Phi(\mathcal{Z})$ is a Kurdyka-Łojasiewicz (KŁ) function.*

Proof. Following [1], [2], since $h(\mathcal{Z})$ is a polynomial function of $N+2$ coupling variables, it is an obviously real-analytic function. Regarding the $f(\mathcal{X})$, since the constraint set onto $\{\mathcal{L} : \mathcal{P}_\Omega(\mathcal{L}) = \mathcal{P}_\Omega(\mathcal{F})\}$ is semi-algebraic, the $\iota(\mathcal{X})$ is a semi-algebraic function resulting from that indicator functions of semi-algebraic sets are semi-algebraic functions. Thus, $\Phi(\mathcal{Z})$, as a finite sum of real-valued analytic and semi-algebraic functions, is a KŁ function. □

Lemma E.2. *Let $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ be the sequence generated by Algorithm 3 of the main text. Then, the sequence $\{\Phi(\mathcal{Z}^{(t)})\}_{t\in\mathbb{N}}$ sufficiently decreases, i.e.,*

$$\Phi(\mathcal{Z}^{(t)}) - \Phi(\mathcal{Z}^{(t+1)}) \geq \frac{\rho}{2}\|\mathcal{Z}^{(t+1)} - \mathcal{Z}^{(t)}\|_F^2, \tag{76}$$

where $\|\mathcal{Z}^{(t+1)} - \mathcal{Z}^{(t)}\|_F^2 = \|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F^2 + \sum_{k=1}^N \|\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}\|_F^2 + \|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F^2$.

Proof. Since the sequences $\{\mathcal{G}_k^{(t)}\}_{t\in\mathbb{N}}$, $k = 1, 2, \cdots, N$, are generated by minimizing

$$\frac{1}{2}\|\mathcal{X}^{(t)} - \text{TW}[\![\mathcal{G}_{1:k-1}^{(t+1)}, \mathcal{G}_k, \mathcal{G}_{k+1:N}^{(t)}; \mathcal{C}^{(t)}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{G}_k - \mathcal{G}_k^{(t)}\|_F^2, \quad k = 1, 2, \cdots, N, \tag{77}$$

each $\mathcal{G}_k^{(t+1)}$ is a minimizer of the optimization problem (77) in $(t+1)$-th iteration, thus leading to inequalities as follows,

$$h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k}^{(t+1)}, \mathcal{G}_{k+1:N}^{(t)}, \mathcal{C}^{(t)}) + \frac{\rho}{2}\|\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}\|_F^2 \leq h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k-1}^{(t+1)}, \mathcal{G}_{k:N}^{(t)}, \mathcal{C}^{(t)}), \ k = 1, 2, \cdots, N. \tag{78}$$

Likewisely, the sequences $\{\mathcal{C}^{(t)}\}_{t\in\mathbb{N}}$ and $\{\mathcal{X}^{(t)}\}_{t\in\mathbb{N}}$ are generated by respectively minimizing

$$\frac{1}{2}\|\mathcal{X}^{(t)} - \text{TW}[\![\mathcal{G}_{1:N}^{(t+1)}; \mathcal{C}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{C} - \mathcal{C}^{(t)}\|_F^2 \tag{79}$$

and

$$\frac{1}{2}\|\mathcal{X} - \text{TW}[\![\mathcal{G}_{1:N}^{(t+1)}; \mathcal{C}^{(t+1)}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{X} - \mathcal{X}^{(t)}\|_F^2 + \iota(\mathcal{X}), \tag{80}$$

thus $\mathcal{C}^{(t+1)}$ and $\mathcal{X}^{(t+1)}$ can respectively satisfy

$$h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) + \frac{\rho}{2}\|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F^2 \leq h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t)}) \tag{81}$$

and

$$h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) + f(\mathcal{X}^{(t+1)}) + \frac{\rho}{2}\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F^2 \leq h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) + f(\mathcal{X}^{(t)}). \tag{82}$$

Especially, if $t \in \{t : t > 200 \text{ and } \texttt{mode}(t, s) \neq 0\}$, then the equal sign in (81) is exactly established owing to $\mathcal{C}^{(t+1)} = \mathcal{C}^{(t)}$. By eliminating the duplicates on the left and right, we can deduce

$$\begin{aligned}
\Phi(\mathcal{Z}^{(t)}) - \Phi(\mathcal{Z}^{(t+1)}) &\geq \frac{\rho}{2}\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F^2 + \sum_{k=1}^N \frac{\rho}{2}\|\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}\|_F^2 + \frac{\rho}{2}\|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F^2 \\
&\geq \frac{\rho}{2}(\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F^2 + \sum_{k=1}^N \|\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}\|_F^2 + \|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F^2) \\
&= \frac{\rho}{2}\|\mathcal{Z}^{(t+1)} - \mathcal{Z}^{(t)}\|_F^2.
\end{aligned} \tag{83}$$

The proof is completed. □

Lemma E.3. *Let $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ be the sequence generated by Algorithm 3 of the main text. Then, there exists*

$$\|\partial\Phi(\mathcal{Z}^{(t+1)})\|_F \le (L_\Phi + (N+2)\rho)\|\mathcal{Z}^{(t+1)} - \mathcal{Z}^{(t)}\|_F, \tag{84}$$

where L_Φ is the sum of the Lipschitz constants of $\partial_{\mathcal{G}_k}\Phi(\mathcal{Z})$ and $\partial_{\mathcal{C}}\Phi(\mathcal{Z})$, i.e., $L_\Phi = \sum_{k=1}^{N} L_{\mathcal{G}_k} + L_{\mathcal{C}}$.

Proof. According to the proof of Lemma E.2, i.e., $\mathcal{G}_k^{(t+1)}$, $k=1,2,\cdots,N$, $\mathcal{C}^{(t+1)}$ and $\mathcal{X}^{(t+1)}$ are the minimum solutions, thus we have $f(\mathcal{X}^{(t+1)}) \equiv 0$ for avoiding $\Phi(\mathcal{Z}) \to \infty$. Especially, when $t > 200$, the variable $\mathcal{C}$ tends to be relatively stable, i.e., $\|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F$ is sufficiently small. Thus, we assume that when $t \in \{t : t > 200 \text{ and } \mathtt{mode}(t,s) \neq 0\}$, $\mathcal{C}^{(j)},\ j = t+2, t+3, \cdots, t+s$, can approximately share the first-order optimal condition of $\mathcal{C}^{(t+1)}$. According to the fact that minimum solutions must satisfy the first-order optimal conditions, i.e., the sub-gradient equations of the objective function, then for all $t \in \mathbb{N}$, we always have

$$\begin{cases} 0 \in \partial_{\mathcal{G}_k} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k}^{(t+1)}, \mathcal{G}_{k+1:N}^{(t)}, \mathcal{C}^{(t)}) + \rho(\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}),\ k = 1,2,\cdots,N, \\ 0 \in \partial_{\mathcal{C}} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) + \rho(\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}), \\ 0 \in \partial_{\mathcal{X}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) + \rho(\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}). \end{cases} \tag{85}$$

Based on the sub-differentiability property, i.e.,

$$\begin{aligned} \partial\Phi(\mathcal{Z}^{(t+1)}) = \big(&\partial_{\mathcal{X}}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \partial_{\mathcal{G}_1}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \partial_{\mathcal{G}_2}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \cdots, \\ &\partial_{\mathcal{G}_N}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \partial_{\mathcal{C}}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\big), \end{aligned} \tag{86}$$

where

$$\begin{cases} \partial_{\mathcal{X}}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) = \partial_{\mathcal{X}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \\ \partial_{\mathcal{G}_k}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) = \partial_{\mathcal{G}_k} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}),\ k=1,2,\cdots,N, \\ \partial_{\mathcal{C}}\Phi(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) = \partial_{\mathcal{C}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}), \end{cases} \tag{87}$$

we have the triangle inequality as follows,

$$\|\partial\Phi(\mathcal{Z}^{(t+1)})\|_F \le \sum_{k=1}^{N} \|\partial_{\mathcal{G}_k} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\|_F + \|\partial_{\mathcal{C}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\|_F + \|\partial_{\mathcal{X}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\|_F.$$

Substitute into the first-order optimal condition (85), leading to

$$\begin{aligned} \|\partial\Phi(\mathcal{Z}^{(t+1)})\|_F \le& \sum_{k=1}^{N} \|\partial_{\mathcal{G}_k} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{G}_k} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k}^{(t+1)}, \mathcal{G}_{k+1:N}^{(t)}, \mathcal{C}^{(t)}) - \rho(\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)})\|_F \\ &+ \|\partial_{\mathcal{C}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{C}} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \rho(\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)})\|_F + \rho\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F \\ \le& \sum_{k=1}^{N} \|\partial_{\mathcal{G}_k} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{G}_k} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k}^{(t+1)}, \mathcal{G}_{k+1:N}^{(t)}, \mathcal{C}^{(t)})\|_F \\ &+ \sum_{k=1}^{N} \rho\|\mathcal{G}_k^{(t+1)} - \mathcal{G}_k^{(t)}\|_F + \|\partial_{\mathcal{C}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{C}} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\|_F \\ &+ \rho\|\mathcal{C}^{(t+1)} - \mathcal{C}^{(t)}\|_F + \rho\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F. \end{aligned} \tag{88}$$

Since $f \equiv 0$ and h is a polynomial function of all factor tensors, Φ is twice continuously differentiable. Moreover, as shown in Lemma E.4, the sequence $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ is bounded. By Lemma E.2, all iterates remain in the bounded level set

$$S := \{\mathcal{Z} : \Phi(\mathcal{Z}) \le \Phi(\mathcal{Z}^{(0)})\},$$

which is compact due to the continuity of Φ. Since the Hessian of Φ is continuous, it is bounded on the set S. Consequently, the partial derivatives $\partial_{\mathcal{G}_k}\Phi(\mathcal{Z})$, $k = 1,\ldots,N$, and $\partial_{\mathcal{C}}\Phi(\mathcal{Z})$ are Lipschitz continuous on the set S, even if not globally. Therefore, there exist positive constants $L_{\mathcal{G}_k}$, $k=1,\ldots,N$, and $L_{\mathcal{C}}$ such that

$$\begin{cases} \|\partial_{\mathcal{G}_k} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{G}_k} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:k}^{(t+1)}, \mathcal{G}_{k+1:N}^{(t)}, \mathcal{C}^{(t)})\|_F \\ \qquad \le L_{\mathcal{G}_k}\|(\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}, \{\mathcal{G}_m^{(t+1)} - \mathcal{G}_m^{(t)}\}_{m=k+1}^{N}, \mathcal{C}^{(t+1)} - \mathcal{C}^{(t)})\|_F,\ k=1,2,\cdots,N, \\ \|\partial_{\mathcal{C}} h(\mathcal{X}^{(t+1)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)}) - \partial_{\mathcal{C}} h(\mathcal{X}^{(t)}, \mathcal{G}_{1:N}^{(t+1)}, \mathcal{C}^{(t+1)})\|_F \le L_{\mathcal{C}}\|\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)}\|_F. \end{cases} \tag{89}$$

Thus, a backsubstitution yields

$$\begin{aligned}\|\partial\Phi(\mathcal{Z}^{(t+1)})\|_F &\leq \sum_{k=1}^{N} L_{\mathcal{G}_k}\|(\mathcal{X}^{(t+1)}-\mathcal{X}^{(t)},\{\mathcal{G}_m^{(t+1)}-\mathcal{G}_m^{(t)}\}_{m=k+1}^{N},\mathcal{C}^{(t+1)}-\mathcal{C}^{(t)})\|_F+\sum_{k=1}^{N}\rho\|(\mathcal{G}_k^{(t+1)}-\mathcal{G}_k^{(t)})\|_F\\ &\quad + L_{\mathcal{C}}\|\mathcal{X}^{(t+1)}-\mathcal{X}^{(t)}\|_F+\rho\|\mathcal{C}^{(t+1)}-\mathcal{C}^{(t)}\|_F+\rho\|\mathcal{X}^{(t+1)}-\mathcal{X}^{(t)}\|_F\\ &\leq (\sum_{k=1}^{N} L_{\mathcal{G}_k}+L_{\mathcal{C}})\|\mathcal{Z}^{(t+1)}-\mathcal{Z}^{(t)}\|_F+(N+2)\rho\|\mathcal{Z}^{(t+1)}-\mathcal{Z}^{(t)}\|_F\\ &= (L_\Phi+(N+2)\rho)\|\mathcal{Z}^{(t+1)}-\mathcal{Z}^{(t)}\|_F,\end{aligned} \tag{90}$$

where $L_\Phi=\sum_{k=1}^{N} L_{\mathcal{G}_k}+L_{\mathcal{C}}$. The proof is completed. □

Lemma E.4. *Let $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ be the sequence generated by Algorithm 3 of the main text, then it is bounded.*

Proof. Relying upon the optimal solution in Algorithm 3 of the main text, i.e.,

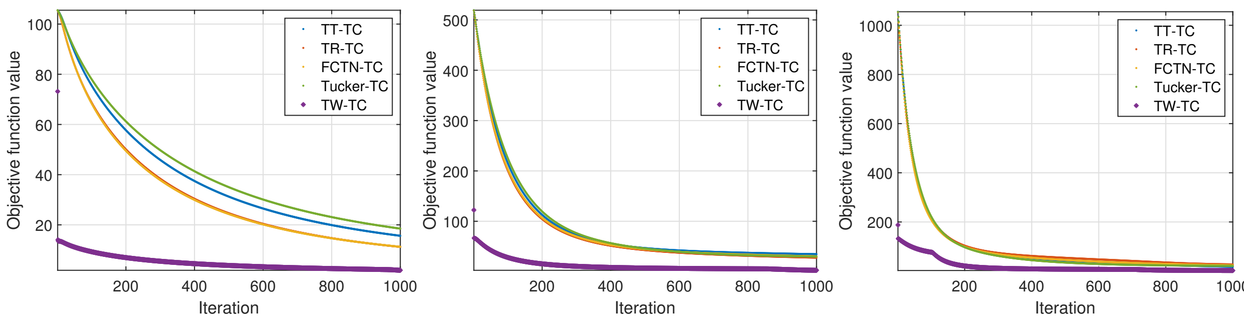
$$\mathcal{X}^{(t+1)}=\mathcal{P}_{\Omega^c}\Big(\frac{\text{TW}[\![\{\mathcal{G}_k^{(t+1)}\}_{k=1}^{N};\mathcal{C}^{(t+1)}]\!]+\rho\mathcal{X}^{(t)}}{1+\rho}\Big)+\mathcal{P}_\Omega(\mathcal{F}), \tag{91}$$

which requires that $\mathcal{X}$ is forcibly projected into set $\{\mathcal{L}:\mathcal{P}_\Omega(\mathcal{L})=\mathcal{P}_\Omega(\mathcal{F})\}$. Thus, the indicator function $\iota(\mathcal{X})\equiv 0$, i.e., $f(\mathcal{X})\equiv 0$ in (75). As shown in Algorithm 3 of the main text, the initial $\mathcal{Z}^{(0)}=(\mathcal{X}^{(0)},\mathcal{G}_{1:N}^{(0)},\mathcal{C}^{(0)})$ is bounded, then $\Phi(\mathcal{Z}^{(0)})$ is bounded. According to Lemma E.2, the sequence $\{\Phi(\mathcal{Z}^{(t)})\}_{t\in\mathbb{N}}$ decreases sufficiently, thus leading to $0\leq\Phi(\mathcal{Z}^{(t)})\leq\Phi(\mathcal{Z}^{(0)})$, i.e., $0\leq h(\mathcal{Z}^{(t)})\leq h(\mathcal{Z}^{(0)})$ for $\forall t\in\mathbb{N}$. Since the continuous function $h(\mathcal{Z})$ is proper and coercive, there exists $\|\mathcal{Z}\|_F\to\infty$ if and only if $h(\mathcal{Z})\to\infty$. Obviously, the discrete points $h(\mathcal{Z}^{(t)})\not\to\infty$, thus $\|\mathcal{Z}^{(t)}\|_F\not\to\infty$, i.e., the sequence $\{\mathcal{Z}^{(t)}\}_{t\in\mathbb{N}}$ is certainly bounded. □

APPENDIX F
NUMERICAL CONVERGENCE

To further investigate the influence of the adopted step-size schedule on the convergence behavior of the proposed TW-TC model, we provide additional empirical analysis in Fig. 1. Specifically, we report the evolution of the objective function values for several tensor completion models under a unified PAM framework, including TT-TC, TR-TC, FCTN-TC, Tucker-TC, and the proposed TW-TC. The convergence experiments are performed on MSI *Beers* from the CAVE dataset. All hyper-parameter settings are configured the same as the main manuscript, and the SRs are fixed at 0.1%, 0.5%, and 1%, respectively. As observed from Fig. 1, all compared models exhibit strictly decreasing objective function values along the iterations, demonstrating stable and monotone convergence behavior under the PAM framework. In particular, the proposed TW-TC model converges within a relatively small number of iterations and rapidly reaches a stable plateau. Notably, although a step-size schedule is adopted for updating the TW factor $\mathcal{C}$ after 200 iterations, no oscillation or degradation in convergence speed is observed. The objective value continues to decrease smoothly and stabilizes shortly thereafter. This indicates that the adopted updating strategy does not adversely affect either the convergence property or the convergence rate in practice.

Overall, the experimental results demonstrate that the intermittent updating strategy serves as a practical acceleration strategy while preserving the convergence stability of the PAM framework. Moreover, compared with other tensor network-based completion models under the same PAM optimization framework, TW-TC achieves competitive or faster empirical convergence behavior, further validating the effectiveness of the proposed model and solver design.



(a) SR = 0.1% (b) SR = 0.5% (c) SR = 1%

Fig. 1. Convergence curves of different tensor completion models on MSI *Beers* from the CAVE dataset under SRs (a) 0.1%, (b) 0.5%, and (c) 1%. Moreover, the iterated numbers of all algorithms are forcibly set to 1000 for all cases without early termination.

REFERENCES

[1] H. Attouch, J. Bolte, and B. F. Svaiter. Convergence of descent methods for semi-algebraic and tame problems: Proximal algorithms, forward–backward splitting, and regularized Gauss–Seidel methods. *Math. Program.*, 137(1):91–129, 2013.

[2] J. Bolte, S. Sabach, and M. Teboulle. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Math. Program.*, 146(1):459–494, 2014.