Tensor Wheel Decomposition and Its Tensor Completion Application

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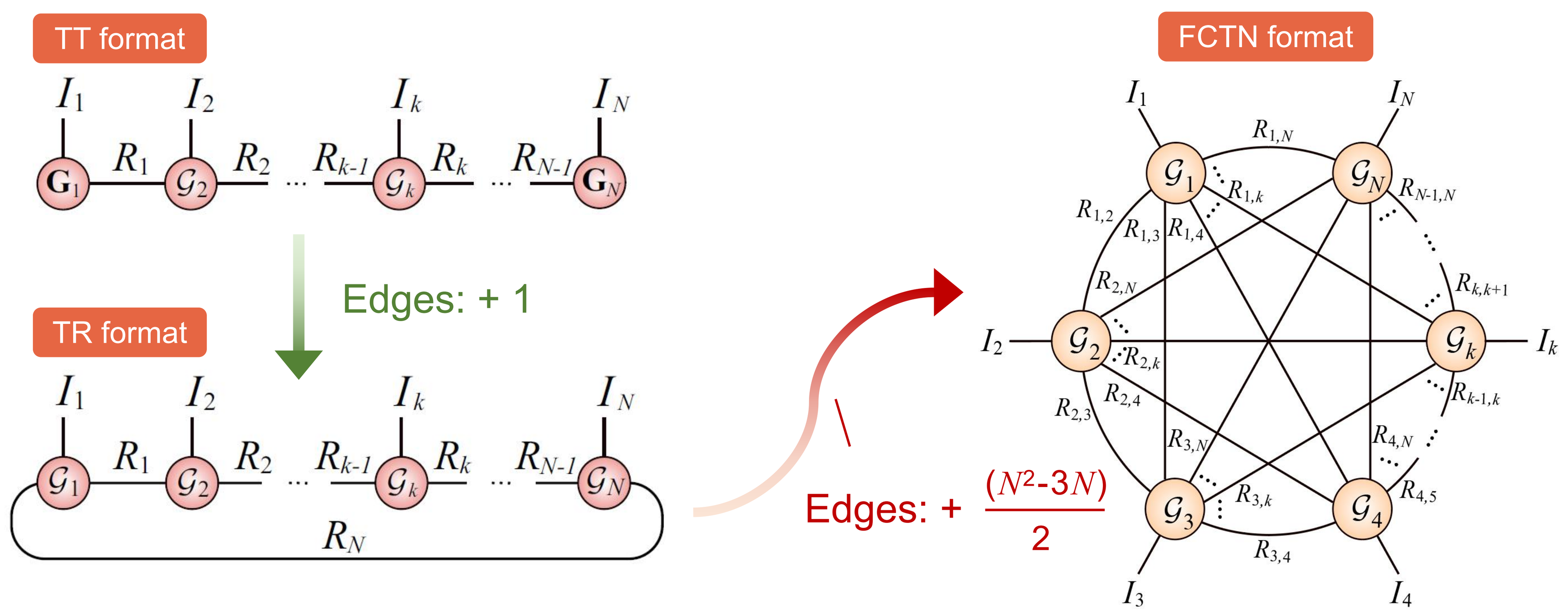
Introduction

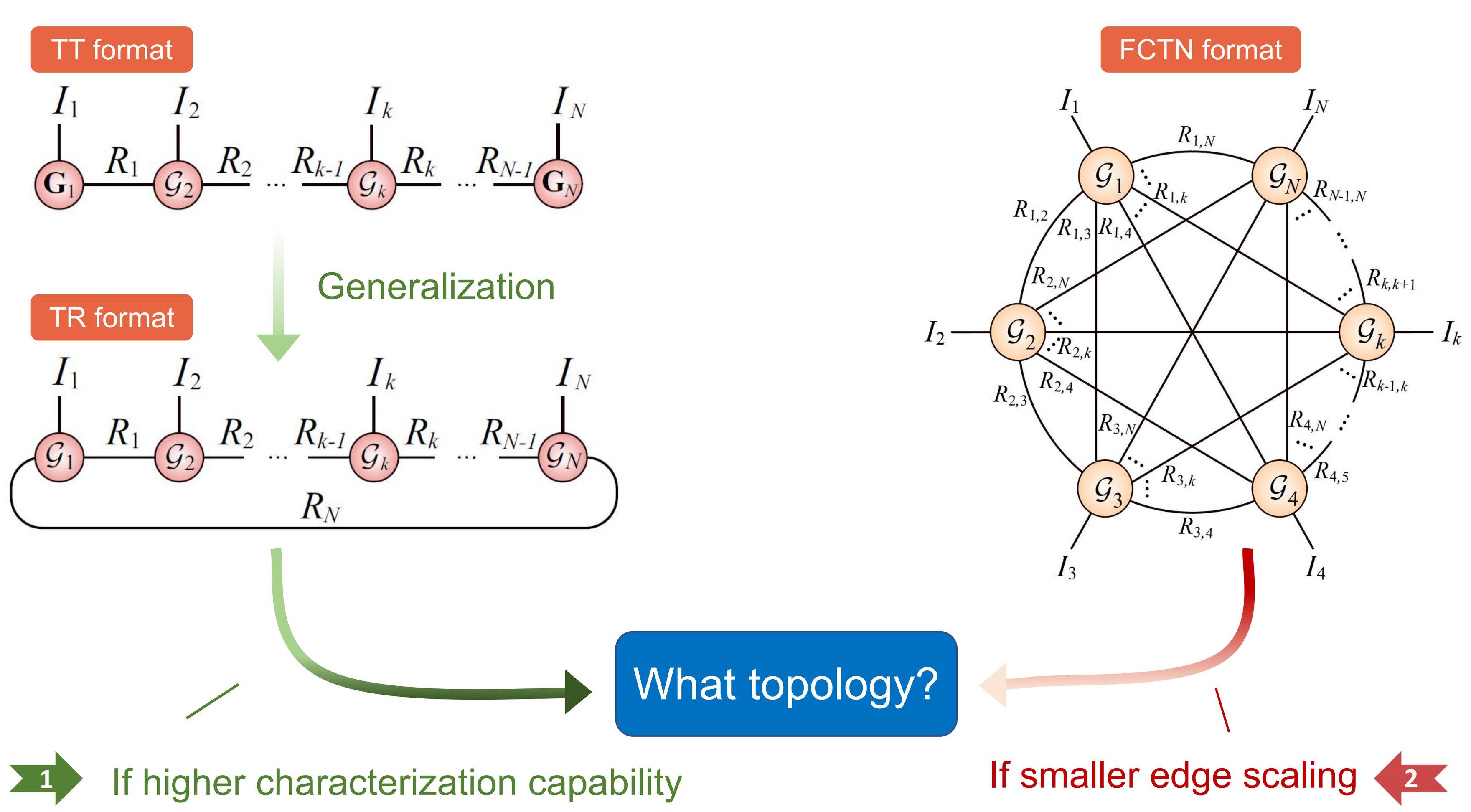
Tensor network (TN) decomposition is prominent for machine learning. However, TN models **developing increasingly sophisticated topology** to pursue incremental gains, which leads to a dramatic increase in rank numbers, thus **encountering laborious hyper-parameter selection**.



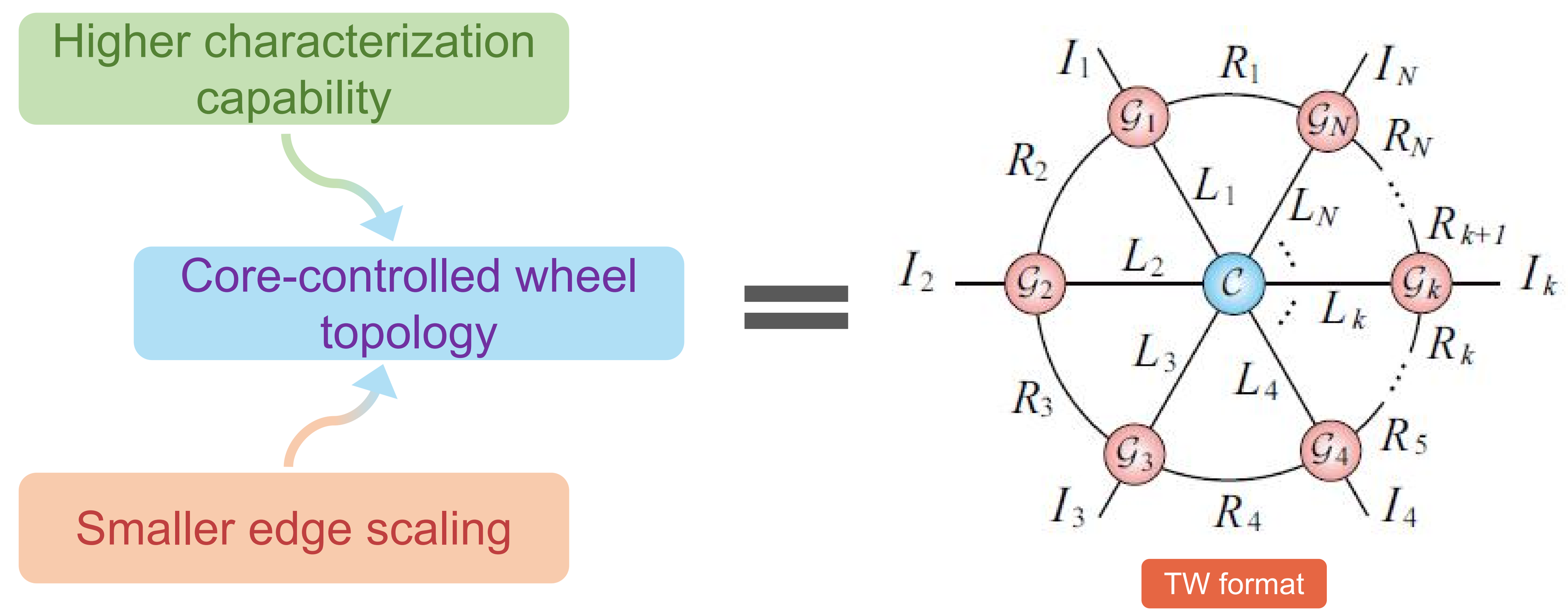
Contributions:

- We analytically **propose a novel TW decomposition**, which allows more expressive characterization for correlation than TT and TR formats, especially preserving the linear scaling for the number of edges with increased tensor dimension.
- We further **provide one numerical application** of TW decomposition, i.e., tensor completion (TC), and then formulate a TW decomposition-based TC model, abbreviated TW-TC.

Motivation



Tensor Wheel Decomposition



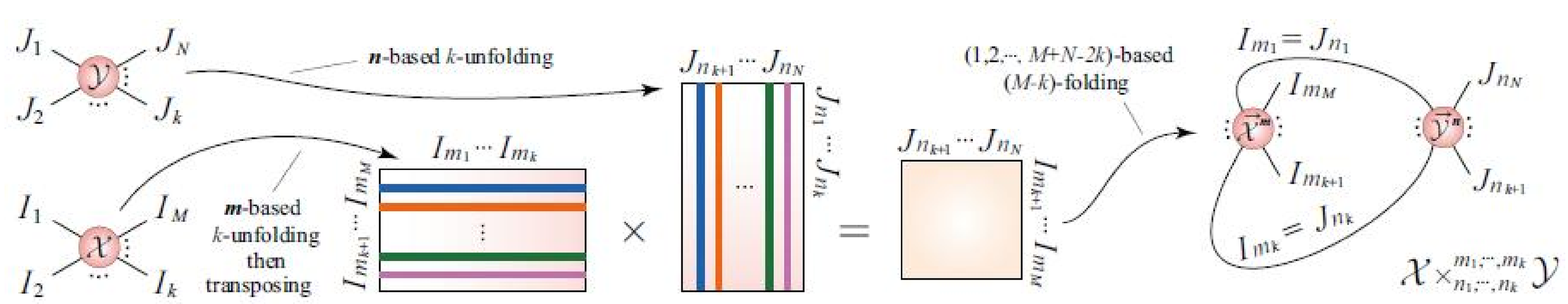
Tensor wheel (TW) decomposition aims to parameterize an Nth-order by both N fourth-order ring factors $\mathcal{G}_k \in \mathbb{R}^{R_k \times I_k \times L_k \times R_{k+1}}, k = 1, 2, \cdots, N$, and an Nth-order core factor $\mathcal{C} \in \mathbb{R}^{L_1 \times L_2 \times \cdots \times L_k \times \cdots \times L_N}$. Mathematically,

- **element-wise relation:**

$$\mathcal{X}(i_1, i_2, \cdots, i_N) = \sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2}\cdots\sum_{r_N=1}^{R_N}\sum_{l_1=1}^{L_1}\cdots\sum_{l_N=1}^{L_N}\{\mathcal{G}_1(r_1, i_1, l_1, r_2)\mathcal{G}_2(r_2, i_2, l_2, r_3)\cdots \mathcal{G}_k(r_k, i_k, l_k, r_{k+1})\cdots\mathcal{G}_N(r_N, i_N, l_N, r_1)\mathcal{C}(l_1, l_2, \cdots, l_N)\},$$

- **tensor-form relation:**

$$\mathcal{X} = \mathcal{G}_1 \times_1^4 \mathcal{G}_2 \times_1^6 \cdots \times_1^{2k} \mathcal{G}_k \times_1^{2k+2} \cdots \times_{1,4}^{2N,1} \mathcal{G}_N \times_{1,2,\cdots,N}^{2,4,\cdots,2N} \mathcal{C}.$$



Numerical Application for Tensor Completion

- **TW-TC model:**

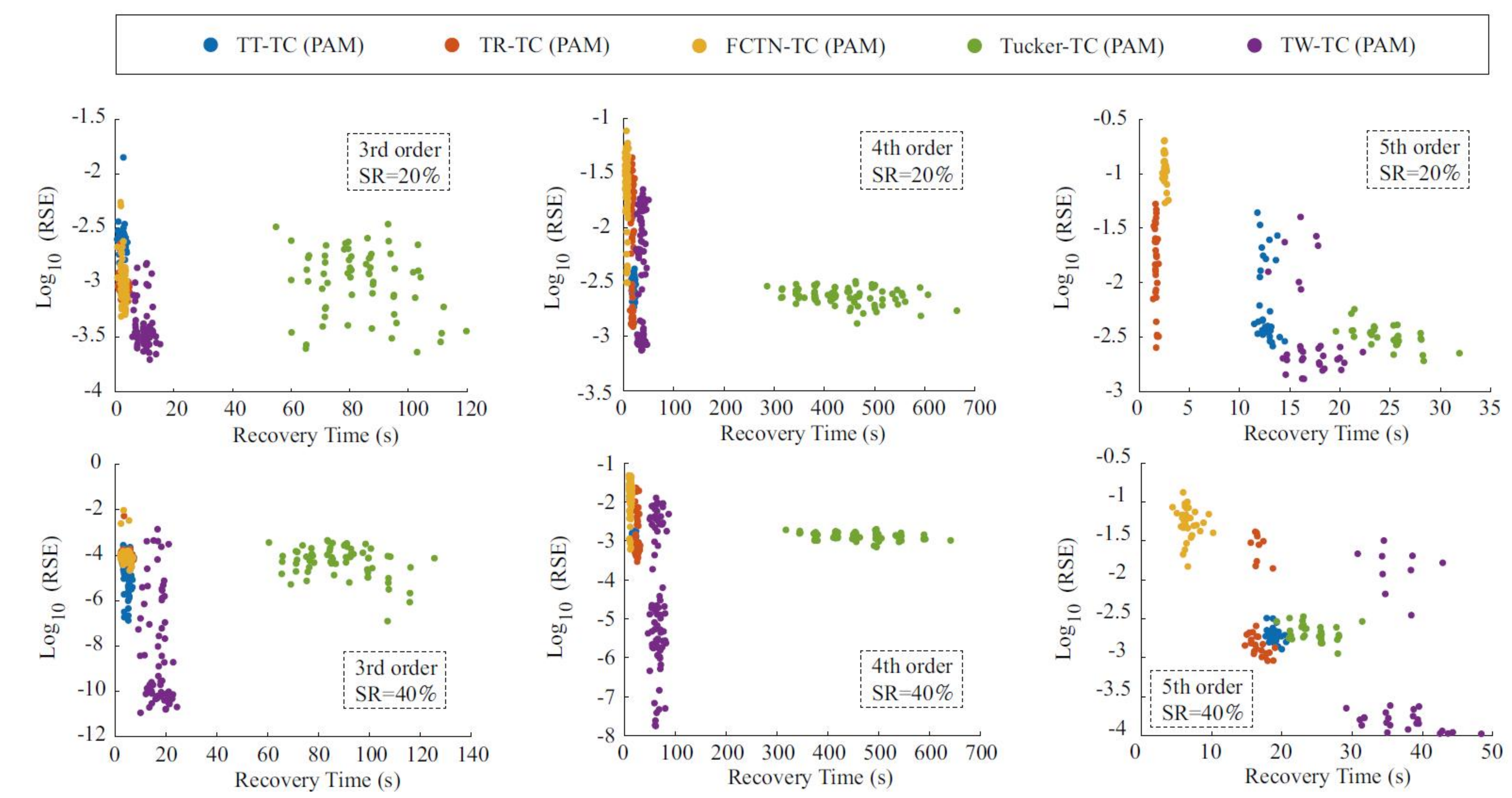
$$\min_{\mathcal{X}, \mathcal{G}_{1:N}, \mathcal{C}} \frac{1}{2}\|\mathcal{X} - \mathrm{TW}[\![\{\mathcal{G}_k\}_{k=1}^N; \mathcal{C}]\!]\|_F^2 + \iota(\mathcal{X}) \text{ with } \iota(\mathcal{X}) := \begin{cases} 0, & \mathcal{X} \in \{\mathcal{L} : \mathcal{P}_\Omega(\mathcal{L}) = \mathcal{P}_\Omega(\mathcal{F})\}; \\ \infty, & \text{otherwise}. \end{cases}$$

- **Iterative algorithm:**

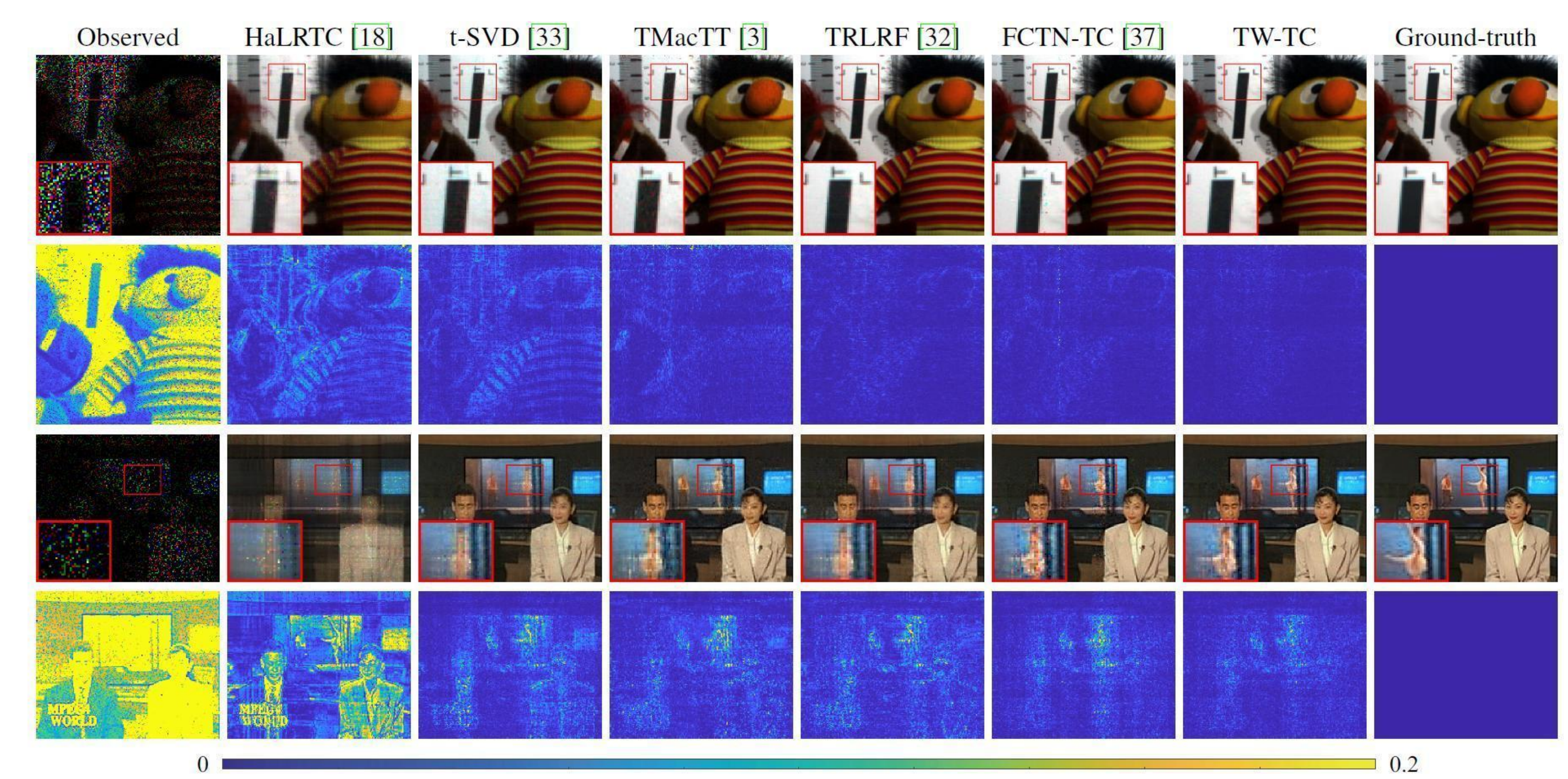
$$\begin{cases} \mathcal{G}_k^{(t+1)} \in \arg\min_{\mathcal{G}_k} \left\{\frac{1}{2}\|\mathcal{X}^{(t)} - \mathrm{TW}[\![\mathcal{G}_{1:k-1}^{(t+1)}, \mathcal{G}_k, \mathcal{G}_{k+1:N}^{(t)}; \mathcal{C}^{(t)}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{G}_k - \mathcal{G}_k^{(t)}\|_F^2\right\}, \\ \mathcal{C}^{(t+1)} \in \arg\min_{\mathcal{C}} \left\{\frac{1}{2}\|\mathcal{X}^{(t)} - \mathrm{TW}[\![\mathcal{G}_{1:N}^{(t+1)}; \mathcal{C}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{C} - \mathcal{C}^{(t)}\|_F^2\right\}, \\ \mathcal{X}^{(t+1)} \in \arg\min_{\mathcal{X}} \left\{\frac{1}{2}\|\mathcal{X} - \mathrm{TW}[\![\mathcal{G}_{1:N}^{(t+1)}; \mathcal{C}^{(t+1)}]\!]\|_F^2 + \frac{\rho}{2}\|\mathcal{X} - \mathcal{X}^{(t)}\|_F^2 + \iota(\mathcal{X})\right\}. \end{cases}$$

Results

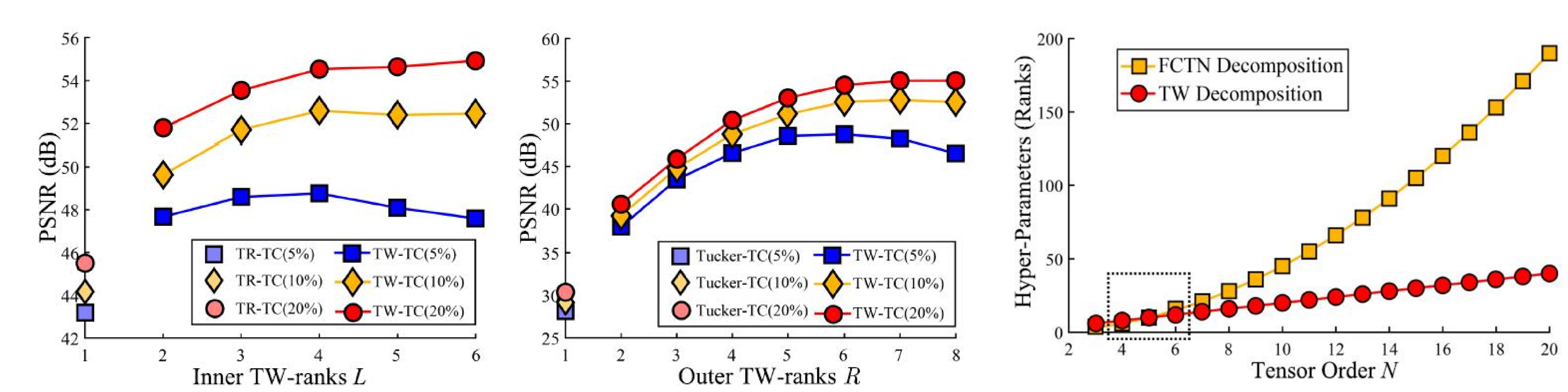
- **Synthetic data completion:**



- **Real-world data completion:**



- **Discussions:**



(a) PSNR versus inner TW-ranks (i.e., $L = L_k$, $k = 1, 2, \cdots, N$) when outer TW-ranks (i.e., $R = R_k$, $k = 1, 2, \cdots, N$) and all TR-ranks are 6.

(b) PSNR versus outer TW-ranks (i.e., $R = R_k$, $k = 1, 2, \cdots, N$) when inner TW-ranks (i.e., $L = L_k$, $k = 1, 2, \cdots, N$) and all Tucker-ranks are 4.

(c) The number of hyper-parameters (i.e., ranks) of FCTN and TW decompositions against tensor dimension.